Chiral Scale and Conformal Invariance in 2D Quantum Field Theory

Diego M. Hofman and Andrew Strominger

Center for the Fundamental Laws of Nature, Harvard University, Cambridge, Massachusetts 02138, USA

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It is well known that a local, unitary Poincaré-invariant 2D quantum field theory with a global scaling symmetry and a discrete non-negative spectrum of scaling dimensions necessarily has both a left and a right local conformal symmetry. In this Letter, we consider a chiral situation beginning with only a left global scaling symmetry and do not assume Lorentz invariance. We find that a left conformal symmetry is still implied, while right translations are enhanced either to a right conformal symmetry or a left $U(1)$ Kac-Moody symmetry.

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Introduction.—A two-dimensional (2D) Poincaré and scale invariant quantum field theory has at least four global symmetries under which the light cone coordinates transform as

$$x^- \to x^- + a, \qquad x^+ \to x^+ + b,$$
$$x^- \to \lambda^- x^-, \qquad x^+ \to \lambda^+ x^+. \tag{1}$$

If, in addition, one posits that the theory is unitary and that the spectrum of the dilation operator for $\lambda^+ = \lambda^-$ is discrete and non-negative, then it was shown in Ref. [1] that the four global symmetries are enhanced to left and right infinite-dimensional conformal symmetries. Explicit counterexamples indicate that the enhancement need not occur if the dilation spectrum is not discrete.

In recent years, interest in 2D quantum field theories with other types of global scaling symmetries has arisen in a variety of contexts ranging from condensed matter to string theory. In this Letter, we consider the special case with three global symmetries:

$$x^- \to x^- + a, \qquad x^+ \to x^+ + b, \qquad x^- \to \lambda x^-, \tag{2}$$

comprising two translational symmetries and a chiral "left" dilational symmetry. Our assumptions will include locality, unitary, and a discrete non-negative dilational spectrum but not Lorentz invariance. In an argument parallel to the one in Ref. [1], we find that these three global symmetries are sufficient to conclude that there are (at least) two infinite-dimensional sets of local symmetries. One of these is a left local conformal symmetry which enhances the left dilational symmetry. The other enhances the right translational symmetry and can be either a right conformal symmetry or a left current algebra.

It is surprising that such a powerful conclusion can be reached from such minimal assumptions. However, the element of surprise is potentially reduced by the fact that at this point there is no definite example of a quantum field theory which nontrivially satisfies all of our stated assumptions. (One possible example might be given by the continuum limit of the large N chiral Potts model, as discussed in Ref. [2].) On the other hand, possible interesting examples are suggested by the recent appearance of warped AdS_3 geometries in a variety of string theoretic investigations including the holographic duals of the so-called dipole deformations of 2D gauge theories [3–8] and the near-horizon geometries of extreme Kerr black holes [9–11]; see also [12–16]. Warped AdS_3 has an $SL(2,R) \times U(1)$ isometry group which contains (2). As these spaces are continuous deformations of AdS_3 spaces with CFT_2 duals, we expect that their holographic duals exist and are deformed CFT_2s with the symmetries (2). However, at present it not clear to what extent these duals obey all the assumptions stated below. In this Letter, we concentrate on the pure field theory analysis and leave these interesting issues to future investigations.

From dilations to Virasoro.—We wish to consider local, unitary, translationally invariant quantum field theories in 2D flat Minkowski space with a (linearly realized) chiral global scale invariance as in (2).

We do not assume Lorentz invariance, an action, or a conserved symmetric stress tensor. The operators generating left-moving (i.e., x^-) translations and dilations will be denoted H and D, respectively, while right-moving translations are denoted $\bar{P}$, where the bar in general denotes right-moving charges. By assumption these operators annihilate the vacuum. Their commutation relations are

$$i[D, H] = H, \qquad i[D, \bar{P}] = 0, \qquad i[H, \bar{P}] = 0. \tag{3}$$

We moreover assume, following Ref. [1], that the eigenvalue spectrum λ_i of D is discrete and non-negative and there exists a complete basis of local operators Φ_i such that

$$i[H, \Phi_i] = \partial_- \Phi_i, \qquad i[\bar{P}, \Phi_i] = \partial_+ \Phi_i,$$
$$i[D, \Phi_i] = x^- \partial_- \Phi_i + \lambda_i \Phi_i, \tag{4}$$

and $\int_C d\Phi_i = 0$ for any closed or complete spacelike contour C. Note that "local operators" as here defined do not involve explicit functions of $x^\pm$. We will refer to λ_i as the weight of the operator Φ_i. Translational plus

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dilational invariance implies the vacuum two-point functions of the $\{\Phi_i\}$ obey

$$\langle \Phi_i(x^-, x^+)\Phi_j(x'^-, x'^+)\rangle = \frac{f_{ij}(x^+ - x'^+)}{(x^- - x'^-)^{\lambda_i+\lambda_j}} \quad (5)$$

for some *a priori* unknown functions f_{ij}.

Noether's theorem implies that each of the operators H, D, and $\bar{P}$ is associated to a conserved Noether current with components denoted $h_\pm$, $d_\pm$, and $p_\pm$ whose dual contour integral, e.g.,

$$H = \int dx^+ h_+ - \int dx^- h_- \quad (6)$$

then gives the global charges. A proof in the present context is reviewed in the appendix. All of these currents have an ambiguity under shifts of the form $\pm\partial_\pm O$, where O is a more general type of operator potentially involving explicit functions of $x^\pm$:

$$O(x^+, x^-) = \sum_i f_i(x^+, x^-)\Phi_i(x^+, x^-). \quad (7)$$

We also show in the appendix that the shifts can be chosen so that the currents satisfy canonical commutation relations, viz.

$$i[H, h_\pm] = \partial_- h_\pm, \qquad i[\bar{P}, h_\pm] = \partial_+ h_\pm, \quad (8)$$

$$i[H, p_\pm] = \partial_- p_\pm, \qquad i[\bar{P}, p_\pm] = \partial_+ p_\pm, \quad (9)$$

$$i[H, d_\pm] = \partial_- d_\pm - h_\pm, \qquad i[\bar{P}, d_\pm] = \partial_+ d_\pm. \quad (10)$$

This implies that $h_\pm$ and $p_\pm$ are local operators, but the term proportional to $h_\pm$ in $i[H, d_\pm]$ implies that $d_\pm$ must have explicit dependence on the x^- coordinate. The appendix demonstrates that the currents can be chosen to be eigenoperators of D. The weights of the global charges (3) then imply

$$\begin{aligned} i[D, h_-] &= x^-\partial_- h_- + 2h_-, \\ i[D, h_+] &= x^-\partial_- h_+ + h_+, \end{aligned} \quad (11)$$

$$i[D, p_-] = x^-\partial_- p_- + p_-, \qquad i[D, p_+] = x^-\partial_- p_+, \quad (12)$$

$$i[D, d_-] = x^-\partial_- d_- + d_-, \qquad i[D, d_+] = x^-\partial_- d_+. \quad (13)$$

We see that d_+ and p_+ are weight 0, d_-, h_+, and p_- are weight 1 and h_- is weight 2. (In an ordinary 2D conformal field theory, d_+, h_+, and p_- all vanish, $p_+ = T_{++}$, $d_- = x^-T_{--}$, and $h_- = T_{--}$.)

Let us now find the explicit coordinate dependence of $d_\pm$ and write the current in terms of local operators. Defining $s_\pm$ by

$$d_\pm = x^- h_\pm + s_\pm \quad (14)$$

and using (8)–(10), we find that

$$i[H, s_\pm] = \partial_- s_\pm, \qquad i[\bar{P}, s_\pm] = \partial_+ s_\pm. \quad (15)$$

We conclude (s_+, s_-) are local operators with weights (0, 1). Conservation of the $d_\pm$ and $h_\pm$ currents imposes

$$h_+ = -\partial_- s_+ - \partial_+ s_-. \quad (16)$$

We have not at this point fixed all the shift freedom in the currents. In particular, we may shift away s_-:

$$h_\pm \to h_\pm \mp \partial_\pm s_-, \qquad d_\pm \to d_\pm \mp \partial_\pm(x^- s_-), \quad (17)$$

which remains consistent with the commutators (8)–(13) as well as current conservation. Equations (14) and (16) now take the simpler form

$$d_+ = x^- h_+ + s_+, \quad d_- = x^- h_-, \quad h_+ = -\partial_- s_+. \quad (18)$$

Now, we can use the general form of the two-point functions given by (5). Bearing in mind the fact that s_+ is a local operator of weight 0, we must have

$$\langle s_+ s_+\rangle = f_{s_+}(x^+), \quad (19)$$

which implies $\partial_- s_+ = h_+ = 0$. Conservation of $h_\pm$ then reduces to $\partial_+ h_- = 0$ or, equivalently, $h_- = h_-(x^-)$. This fact immediately leads to the existence of an infinite set of conserved charges. Define

$$T_\xi = -\int dx^- \xi(x^-) h_-, \quad \bar{J}_\chi = \int dx^+ \chi(x^+) s_+, \quad (20)$$

where $\xi(x^-)$ and $\chi(x^+)$ are smooth functions. In particular, $H = T_1$ and $D = \bar{J}_1 + T_{x^-}$. Notice that, while h_- cannot vanish if we are to have a nontrivial H operator, s_+ could be identically zero. $s_+ \neq 0$ leads to the existence of even more local symmetries unrelated to the originally posited global symmetries. Given that s_+ is independent of x^+ and transforms as a zero weight operator under D, we have

$$i[H, \bar{J}_\chi] = 0, \qquad i[D, \bar{J}_\chi] = 0. \quad (21)$$

The Jacobi identity implies that the commutator of an operator annihilated by H (such as $\bar{J}_\chi$) and a local field (such as h_-) must be a local field itself. Therefore, using (21), we have

$$i[\bar{J}_\chi, h_-] = \partial_-^2 \Phi_\chi, \quad (22)$$

where Φ_χ is a local operator of weight zero. This implies $\partial_- \Phi_\chi = 0$. Immediately, we get

$$i[\bar{J}_\chi, T_\xi] = 0. \quad (23)$$

This means we can minimally set $s_+ = 0$.

The action of $H = T_1$ and $D = T_{x^-}$ on h_- implies

$$i[T_1, T_\xi] = T_{-\xi'}, \quad i[T_{x^-}, T_\xi] = T_{\xi - \xi' x^-}, \quad \xi' \equiv \partial_- \xi. \quad (24)$$

This in turn implies that the action of T_ξ on h_- is

$$i[T_\xi, h_-] = \xi\partial_- h_- + 2\xi' h_- + \partial_-^2 O_\xi. \quad (25)$$

The scaling symmetry plus locality implies O_ξ must be of the form $O_\xi = \xi O_1 + \xi' O_0$ with O_1 a local operator of weight one. As it cannot depend on x^+, O_0 must be a weight zero constant. Integrating both sides with respect to $dx^- \zeta(x^-)$ gives

$$i[T_\xi, T_\zeta] = T_{\xi'\zeta - \zeta'\xi} + \int dx^- \zeta \partial_-^2 O_\xi. \quad (26)$$

Antisymmetry under the exchange of $\xi \leftrightarrow \zeta$ then requires that $O_1 = 0$ and

$$\partial_-^2 O_\xi = O_0 \partial_-^3 \xi. \quad (27)$$

Defining $c = 24\pi O_0$ we end up with the following commutations relations for the charges T_ξ:

$$i[T_\xi, T_\zeta] = T_{\xi'\zeta - \zeta'\xi} + \frac{c}{48\pi} \int dx^- (\xi''\zeta' - \zeta''\xi'). \quad (28)$$

We recognize this as the algebra of the left-moving conformal generators on the Minkowski plane with central charge c.

Enhancement of right-moving translations to a local symmetry.—Notice that up until now we have not made much use of translational invariance in the x^+ dimension. In particular, the above results also apply when we do not possess this symmetry. Let us add $p_\pm$ to the game.

In this case, the key observation is that p_+ is a zero weight local operator, as implied by (8)–(13). This means

$$\langle p_+ p_+ \rangle = f_{p_+}(x^+). \quad (29)$$

Because a Hermitian operator with a vanishing two-point function is trivial, we learn that $\partial_- p_+ = 0$. Current conservation then implies $\partial_+ p_- = 0$. It follows that

$$p_+ = p_+(x^+), \qquad p_- = p_-(x^-). \quad (30)$$

We cannot have both $p_+ = 0 = p_-$ as the charge $\bar{P}$ is generically nonzero, although from what we have seen so far one of them could vanish. We now discuss all possibilities.

$p_- = 0 \Rightarrow$ *right-moving Virasoro algebra.*—In this case we have infinitely many charges given by

$$\bar{T}_\xi = \int dx^+ \xi(x^+) p_+. \quad (31)$$

Since $\bar{T}_1 = \bar{P}$, we have

$$i[\bar{T}_1, \bar{T}_\xi] = -\bar{T}_{\xi'}. \quad (32)$$

This, in turn, constraints the action of $\bar{T}_\xi$ on p_+ to be

$$i[\bar{T}_\xi, p_+] = \xi \partial_+ p_+ + 2\xi' p_+ + \partial_+ \bar{O}_\xi. \quad (33)$$

If we compare this expression with (25), we see that we are very close to the previous situation for T_ξ. Multiplying by $\zeta(x^+)$ and integrating both sides of this equation, we get

$$i[\bar{T}_\xi, \bar{T}_\zeta] = \bar{T}_{\xi\zeta' - \xi'\zeta} + \int dx^+ \zeta \partial_+ \bar{O}_\xi. \quad (34)$$

Antisymmetry with respect to exchange of ξ and ζ then implies that $\bar{O}_\xi$ is an even number of derivatives of ξ. The term with no derivatives can be eliminated by a constant shift of p_+. Terms with four or more derivatives would violate the Jacobi identity. We conclude (shifting p_+ by a constant if needed)

$$i[\bar{T}_\xi, \bar{T}_\zeta] = \bar{T}_{\xi'\zeta - \zeta'\xi} + \frac{\bar{c}}{48\pi} \int dx^+ (\xi''\zeta' - \zeta''\xi'). \quad (35)$$

We recognize this as the algebra of the right-moving conformal generators on the Minkowski plane with central charge $\bar{c}$. (Interestingly, it is this right-moving Virasoro that gives the entropy in Kerr-CFT [9].)

Of course, the vacuum will not in general be invariant under the global $SL(2,R)_R$ subgroup. Acting with D and H on $\bar{T}_\xi$ we can check that $i[\bar{T}_\xi, h_-] = \partial_-^2 \Phi_\xi$. But Φ_ξ must be a weight zero operator, so we are left with $i[\bar{T}_\xi, h_-] = 0$. The upshot is that $[\bar{T}_\xi, T_\zeta] = 0$, as expected.

$p_+ = 0 \Rightarrow$ *left-moving current algebra.*—In this case we have infinitely many left-moving charges

$$J_\chi = -\int dx^- \chi(x^-) p_-(x^-). \quad (36)$$

Because the zero mode J_1 acts as ∂_+, we must have $i[J_1, p_-] = 0$. This implies $[J_1, J_\chi] = 0$ and hence

$$i[J_\chi, p_-] = \partial_- M_\chi, \quad (37)$$

where the operator M_χ is, by locality, a linear function of χ. Now we are in a position to repeat the arguments used around (25). Multiplying by $\psi(x^-)$, integrating over x^-, and invoking antisymmetry and the Jacobi identity, we find

$$i[J_\chi, J_\psi] = \frac{k}{4\pi} \int dx^- (\psi'\chi - \chi'\psi), \quad (38)$$

where the constant k parameterizes the central element. This is a $U(1)$ Kac-Moody current algebra.

We also need the $[T_\xi, J_\chi]$ commutator. The fact that $[J_1, T_\xi] = 0$ implies

$$[T_\xi, p_-] = \xi \partial_- p_- + \xi' p_- + \partial_- N_\xi \quad (39)$$

with the operator N_ξ linear in ξ. The Jacobi identity with a third operator T_ζ then implies $N_\xi = f \partial_- \xi$ for some constant f [17,18]. If f is nonzero, the current p_- is not a dimension one chiral current. However, we may then shift h_- by a k-dependent multiple of $\partial_- p_-$ so that p_- is a good dimension one current. This shift affects the central charge of T_ξ. Performing this transformation leaves us with the standard commutator

$$i[T_\xi, J_\chi] = J_{-\xi\chi'}. \quad (40)$$

It may seem rather strange to have a left-moving Kac-Moody current algebra whose zero mode generates right translations. However, reminiscent structures have appeared before. In the Kaluza-Klein circle reduction of

AdS_3 to AdS_2, one begins with two Virasoros and ends with a single left-moving Virasoro and current algebra associated to the Kaluza-Klein $U(1)$ [19]. The left current algebra zero mode J_0 descends from the right Virasoro zero mode in AdS_3. Related structures have appeared in Kerr-CFT, where left Virasoro and right current algebra zero modes are sometimes identified [20], as well as in the study of asymptotic symmetries of warped AdS_3 [21].

Nonminimal $p_- \neq 0$, $p_+ \neq 0$.—In this case left and right currents decouple. The commutators $i[\bar{P}, \bar{T}_\xi] = \bar{T}_{-\xi'}$ implies $i[\bar{T}_\xi, p_-] = \partial_- \Phi_\xi$, for some local Φ_ξ. Furthermore, because $\bar{T}_\xi$ does not transform under D and p_- is a weight 1 operator, Φ_ξ must be weight 0. Therefore

$$i[\bar{T}_\xi, J_\chi] = 0, \tag{41}$$

implying that the conserved charges can be analyzed separately as above.

In summary, left translational and dilational symmetries together with right translations imply the existence of (at least) two sets of infinite-dimensional algebras. On the left we always find a local conformal symmetry, while the right translational current is enhanced either to a local right conformal symmetry or a left $U(1)$ current algebra.

Noether's theorem.—Here we will prove Noether's theorem for H and P and put the corresponding currents in canonical "diagonal" form. We assume the existence of a unitary Hamiltonian $\mathcal{H}$ whose commutator with any operator obeys

$$i[\mathcal{H}, O] = \frac{dO}{dt} - \frac{\partial O}{\partial t}, \tag{42}$$

where the last derivative acts on any explicit coordinate dependence in O. Conserved charges are defined as any operator $\mathcal{Q}$ such that $\frac{d\mathcal{Q}}{dt} = 0$. Locality implies

$$\mathcal{Q} = \int_{-\infty}^{\infty} dx q_t(x, t). \tag{43}$$

Charge conservation is then

$$\frac{d\mathcal{Q}}{dt} = \int dx \frac{q_t}{dt} = 0, \tag{44}$$

implying $\frac{q_t}{dt} = -\frac{q_x}{dx}$, for some q_x. (q_x, q_t) is the sought after conserved Noether current associated to $\mathcal{Q}$.

We further assume the existence of a conserved momentum charge $\mathcal{P}$ commuting with $\mathcal{H}$ and obeying

$$i[\mathcal{P}, O] = \frac{dO}{dx} - \frac{\partial O}{\partial x} \tag{45}$$

for any operator O. From these we construct left and right translation charges $2H = \mathcal{H} - \mathcal{P}$ and $2\bar{P} = \mathcal{H} + \mathcal{P}$. If a conserved charge $\mathcal{Q}$ commutes with both H and $\bar{P}$, the associated Noether current must obey

$$i[H, q_\pm] = \partial_- q_\pm \pm \partial_\pm F, \qquad i[\bar{P}, q_\pm] = \partial_+ q_\pm \pm \partial_\pm G, \tag{46}$$

where $\partial_\pm$ are total derivatives with respect to $x^\pm = t \pm x$ and F and G can be expanded

$$F = \sum_i f_i(x^+, x^-)\Phi_i, \qquad G = \sum_i g_i(x^+, x^-)\Phi_i. \tag{47}$$

The Jacobi identity relates the coefficients f_i and g_i as an integrability condition implying the existence of a set of functions r_i such that $g_i = \partial_+ r_i$ and $f_i = \partial_- r_i$. Now let us use the shift freedom $q_\pm \rightarrow q_\pm \mp \partial_\pm R$, with $R = \sum_i r_i(x^+, x^-)\Phi_i$. We then obtain the canonical form of the commutators

$$i[H, q_\pm] = \partial_- q_\pm, \qquad i[\bar{P}, q_\pm] = \partial_+ q_\pm. \tag{48}$$

In particular, this applies to $q_\pm = h_\pm, p_\pm$. Notice that, for any local (coordinate independent) operator Φ, we can still shift $q_\pm \rightarrow q_\pm \mp \partial_\pm \Phi$ and preserve the above commutation relations.

Now we show, following Ref. [1], that the currents can also be made dilation eigenoperators. Current conservation and the commutation relations (3) imply

$$\begin{aligned} i[D, h_\pm] &= x^- \partial_- h_\pm + \lambda(h_\pm) h_\pm \pm \partial_\pm O_h, \\ i[D, p_\pm] &= x^- \partial_- p_\pm + \lambda(p_\pm) p_\pm \pm \partial_\pm O_p, \end{aligned} \tag{49}$$

where $\lambda(p_+, p_-, h_+, h_-) = (0, 1, 1, 2)$. The Jacobi identity can be used to show that O_h and O_p are local operators with no explicit coordinate dependence. This means they are expandable in the discrete basis (4) as $O_q = \sum_i a_i \Phi_i$ for $q = h, p$, where Φ_i has weight λ_i. Let us now shift

$$q_\pm \rightarrow q_\pm \mp \partial_\pm \sum_i b_i \Phi_i, \qquad b_i = \frac{a_i}{w(q_+) - \lambda_i}, \tag{50}$$

for $\lambda_i \neq w(q_+)$. This shift eliminates all the Φ_i in O_q with weights $\lambda_i \neq w(q_+)$. Operators with weight equal to $w(q_+)$ cannot appear in O_q by the assumption (4) that the spectrum of D is discrete and diagonalizable. We are therefore left with the canonical form of the commutators:

$$\begin{aligned} i[D, h_\pm] &= x^- \partial_- h_\pm + \lambda(h_\pm) h_\pm, \\ i[D, p_\pm] &= x^- \partial_- p_\pm + \lambda(p_\pm) p_\pm. \end{aligned} \tag{51}$$

A slight variant of the preceding arguments can be used on the current $d_\pm$ associated to D to set

$$i[H, d_\pm] = \partial_- d_\pm - h_\pm, \qquad i[H, d_\pm] = \partial_+ d_\pm, \tag{52}$$

$$i[D, d_\pm] = x^- \partial_- d_\pm + \lambda(d_\pm) d_\pm, \tag{53}$$

where $\lambda(d_+, d_-) = (0, 1)$. We note that demanding that the current or charge commutators take this canonical form does not fix all the ambiguity in the former. Some of the remaining shift freedom is exploited here, around Eq. (17), to shift away s_-.

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