Experimental Realization of an Approximate Partial Transpose for Photonic Two-Qubit Systems

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We report the first experimental realization of an approximate partial transpose for photonic two-qubit systems. The proposed scheme is based on the local operation on single copies of quantum states and classical communication, and therefore can be easily applied for other quantum information tasks within current technologies. Direct detection of entanglement, i.e., without performing quantum state tomography, using the partial transpose operation, is also demonstrated.

Introduction.—Entanglement is generally a resource for quantum information applications [1,2]. Given composite quantum systems for such applications, it is then required to determine if they are in entangled states rather than to identify their quantum states, e.g., via quantum state tomography. This naturally defines the so-called direct detection of entanglement [3]. As first pointed out by Peres [4], the entanglement of quantum states can be detected by performing the partial transpose (PT) operation on the composite quantum system; i.e., the quantum state of one subsystem is left untouched while the quantum state of the other subsystem is transposed. However, PT is a nonphysical operation as it does not preserve physical symmetries [5]. The impossibility of directly applying the PT operation in experiments has lead to the development of the alternative entanglement detection method based on measuring local observables known as entanglement witnesses [6].

Direct detection of entanglement using PT started to attract interest when Horodecki and Ekert proposed a method called structural physical approximation (SPA) by which nonphysical operations such as PT can be systematically approximated by physical operations [7]. Moreover, it has been shown that SPAs to the nonphysical operations (including PT) can be factorized into local operations and classical communication (LOCC) [8]. Being based on applications of operations rather than observables, entanglement detection using SPAs works with no dependence on a local basis of given quantum states. Using the SPA to PT (SPA-PT), therefore, all entangled states of two qubits as well as other useful entangled states in high dimensions can be detected [9].

Therefore, for an operation-based approach to the direct detection of entanglement [7], it is of utmost importance to devise a practical SPA-PT scheme and demonstrate its feasibility towards entanglement detection. We also emphasize from the fundamental point of view that, realizing nonphysical operations (i.e., not allowed in quantum theory) in their approximate and optimal forms would characterize and confirm how far one can go in manipulating

quantum states for information tasks within the fundamental limit.

Experimental implementation of SPA-PT, on the other hand, has not been successful to date as the original proposal of Horodecki and Ekert requires quantum memory and collective measurement, both of which are far from mature technologies. Very recently however, it has been shown that SPAs to optimal positive maps (including SPA-PT [10]) can, in general, be replaced by quantum channels of a local measurement followed by the preparation of quantum states [11]. This, in fact, significantly improves the experimental feasibility of the direct detection of entanglement using SPAs within present-day technology.

So far, apart from the experimental feasibility *per se*, little has been known about how to carry out the SPA-PT in practice with minimal experimental resources. In this Letter, we provide and demonstrate a practical scheme to realize the SPA-PT for two-qubit states based solely on local measurements and classical communication. The experimental demonstration of the SPA-PT scheme is performed in photonic systems, i.e., using single-photon polarization qubits and linear optical devices. The results show that the SPA-PT scheme works equally well for all Bell-states, indicating no dependence on local basis, a crucial feature for direct detection of entanglement. Direct detection of entanglement, i.e., without performing quantum state tomography, using SPA-PT is also demonstrated.

Scheme.—Let us begin by describing the theoretical scheme to realize the SPA-PT. The central idea of the SPA to a linear map Λ lies in the fact that by admixing with the depolarization, $D[\rho] = I_d/d$ where I_d is the identity matrix in a d dimension, the map Λ can be transformed to a completely positive map $\tilde{\Lambda}$. The SPA map $\tilde{\Lambda} = (1 - p)\Lambda + pD$ with the minimum p ($0 \leq p \leq 1$) represents a physical operation [12]. For linear maps $\mathbb{1} \otimes \Lambda$ that can detect entangled states, the SPA then works as $\widetilde{\mathbb{1} \otimes \Lambda} = (1 - p)\mathbb{1} \otimes \Lambda + pD \otimes D$ with the minimum p ($0 \leq p \leq 1$) [7] and moreover can be factorized into a

form of LOCC [8]. Having collected all these facts and applied them to the case of the PT, one can derive the following decomposition for the SPA-PT for a two-qubit state ρ_{AB} [13]:

$$(\widetilde{\mathbb{1} \otimes T})[\rho_{AB}] = \tfrac{1}{3}(\mathbb{1} \otimes \tilde{T})[\rho_{AB}] + \tfrac{2}{3}(\tilde{\Theta} \otimes D)[\rho_{AB}], \quad (1)$$

where $\tilde{T}$ and $\tilde{\Theta}$ denote SPAs to the transpose and to the inversion, respectively, where the inversion $\Theta[\rho] = -\rho$.

Let us now find those local operations that compose of the SPA-PT. First, it is shown in Ref. [11] that the SPA transpose $\tilde{T}$ corresponds to a channel based on the measurement and preparation of quantum states. It is also shown that the operation can be constructed explicitly as [14] $\tilde{T}[\rho] = \sum_{k=1}^{4} \text{tr}[M_k \rho]|v_k\rangle\langle v_k|$ for a state ρ, where

$$|v_1\rangle \propto |0\rangle + \frac{ie^{i\pi 2/3}}{i + e^{-i\pi 2/3}}|1\rangle,$$

$$|v_2\rangle \propto |0\rangle - \frac{ie^{i\pi 2/3}}{i - e^{-i\pi 2/3}}|1\rangle,$$

$$|v_3\rangle \propto |0\rangle + \frac{ie^{i\pi 2/3}}{i - e^{-i\pi 2/3}}|1\rangle,$$

$$|v_4\rangle \propto |0\rangle - \frac{ie^{i\pi 2/3}}{i + e^{-i\pi 2/3}}|1\rangle,$$

and $\{M_k = |v_k^*\rangle\langle v_k^*|/2\}_{k=1}^{4}$ is a complete measurement. Next, a channel corresponding to $\tilde{\Theta}$ can be constructed using its Choi-Jamiolkowski state $\rho_{\tilde{\Theta}} = [\mathbb{1} \otimes \tilde{\Theta}](|\phi^+\rangle \times \langle\phi^+|)$ where $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Then, $\tilde{\Theta}[\rho] = 2\,\text{tr}_A[\rho_{\tilde{\Theta}}(\rho^T \otimes \mathbb{1})]$ [15]. One easily finds that $\rho_{\tilde{\Theta}}$ is separable, and moreover it holds that $\rho_{\tilde{\Theta}} = (\mathbb{1} \otimes \sigma_y)\rho_{\tilde{T}}(\mathbb{1} \otimes \sigma_y)$ with the Pauli matrix σ_y, where $\rho_{\tilde{T}}$ denotes the Choi-Jamiolkowski state of the channel $\tilde{T}$. Consequently, the channel $\tilde{\Theta}$ is the composition of $\tilde{T}$ and σ_y: $\tilde{\Theta}[\cdot] = \sigma_y \tilde{T}[\cdot]\sigma_y$, i.e., for a qubit state ρ

$$\tilde{\Theta}[\rho] = \sum_{k=1}^{4} \text{tr}[M_k\rho]\sigma_y|v_k\rangle\langle v_k|\sigma_y. \quad (2)$$

That is, the SPA inversion applies measurements that are the same with those in the SPA Transpose while the Pauli operation σ_y is applied in a state preparation.

Realization.—The SPA-PT in Eq. (1) can be implemented by applying $\mathbb{1} \otimes \tilde{T}$ and $\tilde{\Theta} \otimes D$ with probabilities $1/3$ and $2/3$, respectively, see Fig. 1(a). The local operations are actually (single-copy) measurements followed by state preparation, see Fig. 1(b) and 1(c). The measurement M_k, i.e., on the basis $|v_k^*\rangle = U_k|0\rangle$, is performed by unitary transformation U_k and a measurement in the computational basis. The preparation step can be done by transforming a state collapsed by measurement, to a corresponding one using optical elements [14]. For photonic polarization qubits, $|0\rangle = |H\rangle$ and $|1\rangle = |V\rangle$, wave plates and a polarizer are optical elements to perform unitary transformations and the measurement in the computational basis, respectively. If a single photon is found after passing
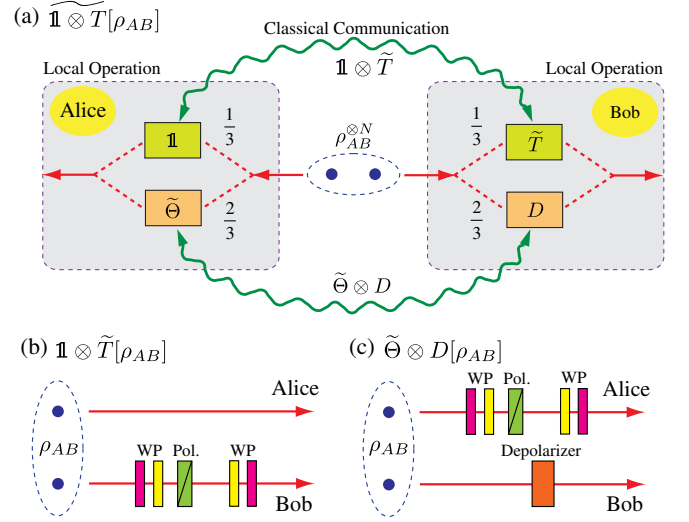

FIG. 1 (color online). The SPA-PT $\widetilde{\mathbb{1} \otimes T}$ for the input state ρ_{AB} is accomplished by applying $\mathbb{1} \otimes \tilde{T}$ and $\tilde{\Theta} \otimes D$ with probabilities $1/3$ and $2/3$, respectively, shown in (a). Combinations of wave plates (WP) and polarizers (Pol.) implement measurement and preparation of quantum states, shown in (b) and (c). See text for details.

through a set of wave plates for U_k and a polarizer aligned for measurement in $|0\rangle\langle 0|$, it would mean that the qubit has collapsed to the state $|0\rangle$ due to the measurement. The collapsed state $|0\rangle$ would then be used for state preparation according to $\tilde{T}$ or $\tilde{\Theta}$, see Figs. 1(b) and 1(c). The depolarization D can be performed by random applications of Pauli matrices, $D[\cdot] = 1/4\sum_{i=0,x,y,z}\sigma_i[\cdot]\sigma_i$, with each σ_i implemented by wave plates.

The details for the experimental realization are the following. A two-qubit state is prepared using the spontaneous parametric down-conversion process. A 6 mm thick type-I β-BaB$_2$O$_4$ crystal is used in the frequency-degenerate, noncollinear phase matching condition. The β-BaB$_2$O$_4$ crystal was pumped by a 405 nm diode laser beam (100 mW) and the spontaneous parametric down-conversion photon pairs were centered at 810 nm. All four two-qubit Bell states (i.e., two-photon polarization Bell states) were then prepared with quantum interferometry [16]. We observed the coincidence counting rate of approximately 900 Hz using interference filters with full width at half-maximum bandwidth of 5 nm. To perform the SPA-PT, each set of measurements and preparations is switched every 5 sec, and the whole operation is also repeated 3 times.

Let us now demonstrate the SPA-PT scheme for Bell states $|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$, $|\psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. These states are particularly chosen to show that the performance depends largely on correlations existing in quantum states but not their local basis. This is indeed an important feature toward the efficient detection of entanglement. In experiments, the initially prepared Bell states and the resulting states after the SPA-PT are identified

using quantum state tomography (QST) [17]. The experimental results are shown in Fig. 2.

To quantify the performance, we compute the Uhlmann's fidelity $F(\rho, \sigma) = [\mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}]^2$ between two states, one

from the experimental realization $(\mathbb{1} \widetilde{\otimes} T)_{\mathrm{exp}}$ and the other from the ideal one $\mathbb{1} \widetilde{\otimes} T$ [18]. For the states $|\phi^{\pm}\rangle$, $|\psi^{-}\rangle$, our experiment shows $F = 0.999$, and for $|\psi^{+}\rangle$, $F = 0.998$. These results show that for the proposed
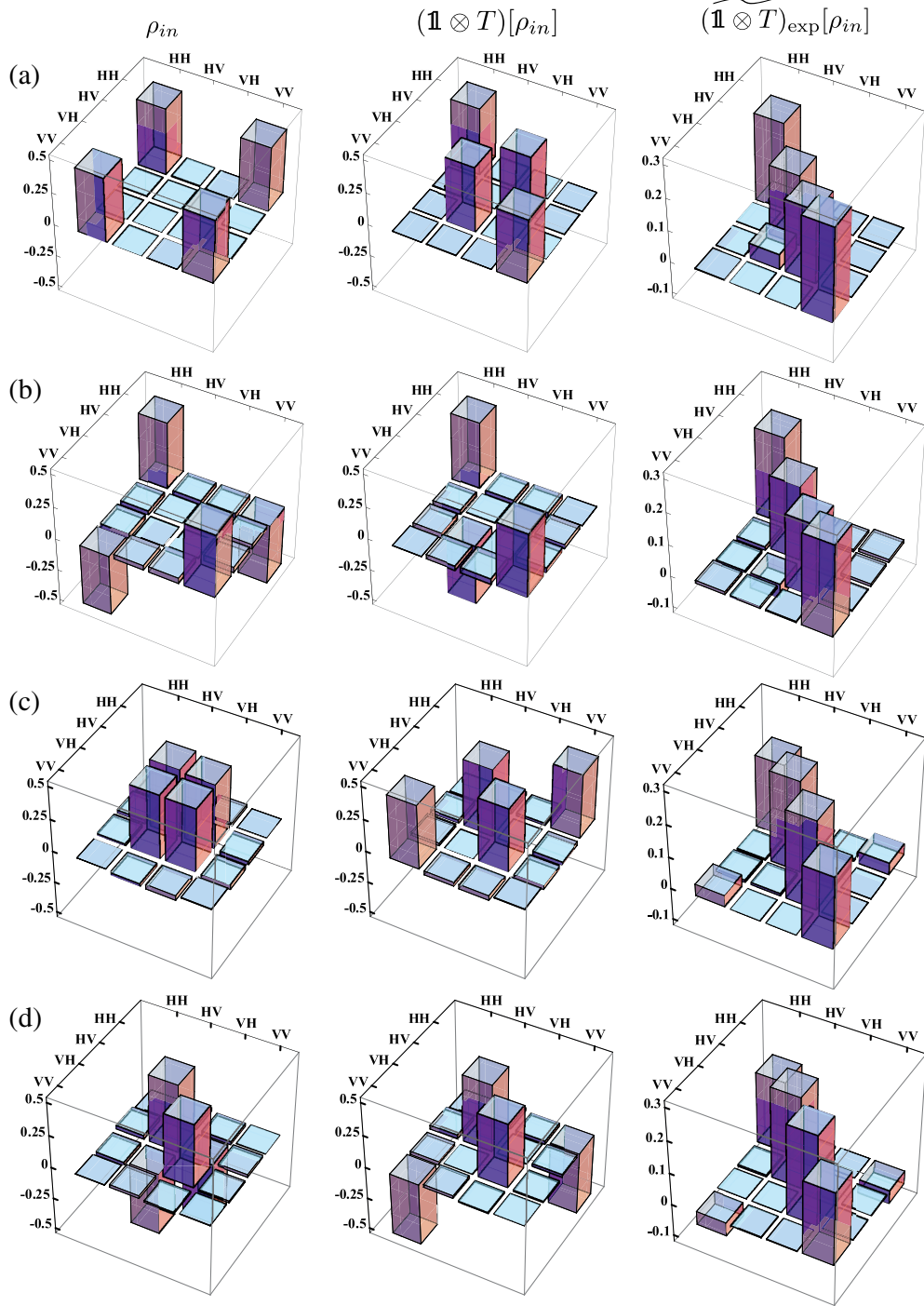

FIG. 2 (color online).   The 1st column shows the QST of the four Bell states prepared in experiments ρ_{in}: (a) $|\phi^{+}\rangle$, (b) $|\phi^{-}\rangle$, (c) $|\psi^{+}\rangle$, and (d) $|\psi^{-}\rangle$. The resulting states after applying the SPA-PT are identified by QST, which is shown in the 3rd column. Note that only the real parts of the density matrices are shown as the imaginary parts are almost zero. These can be compared with the ideal case of the PT, shown in the 2nd column.

SPA-PT scheme only based on LOCC, the experimental realization works faithfully without a dependence on local basis.

Application.—So far, a scheme to realize the SPA-PT based on a single-copy measurement and preparation of quantum states is proposed and its experimental realization is shown. An important application of the SPA-PT, for which it was originally proposed, is the entanglement detection of unknown quantum states [7]. Applying the PT to a two-qubit state, a negative eigenvalue is a sufficient condition to conclude that the state is entangled. When applying the SPA-PT, the condition would be an eigenvalue smaller than the portion of admixed noise. It is an eigenvalue found to be smaller than 2/9 for two-qubit states [7]. Thus, once the SPA-PT is applied to unknown two-qubit states, to determine if they are entangled requires us to estimate the minimum eigenvalues of resulting quantum states [13].

It has been known that a general method to obtain the minimum eigenvalue of unknown quantum states requires a collective measurement within which one should be able to store quantum states for a while [19]. Since the SPA-PT is now performed by local measurements, the spectrum estimation no longer requires a collective measurement on resulting quantum states. This simplifies experimental resources and thus hugely improves the practical feasibility [11]. Estimating eigenvalues of resulting states after SPA-PT then defines a classical optimization problem over measurement outcomes.

Here, the goal is a proof-of-principle demonstration for detecting entanglement (of unknown quantum states) from measurement outcomes of the SPA-PT only. To this end, we consider a "brute-force" approach in the following, in the sense that the cost, such as measurement settings, is not optimized in terms of its efficiency at this stage. This means that all measurement outcomes are collected from $\tilde{T}$, $\tilde{\Theta}$, and D and applied to determine if given quantum states are entangled or separable. For the identity operation in $\mathbb{1} \otimes \tilde{T}$, any measurement in a tomographically complete basis $\{|t_i\rangle\}_{i=1}^4$ is applied. In experiments, measurements of $\mathbb{1} \otimes \tilde{T}$ and, similarly, measurements of $\tilde{\Theta} \otimes D$ are repeated for the two-qubit state ρ_{AB}. The probabilities are then obtained from these measurement outcomes: $p_{ij} = \text{tr}[\rho_{AB}|t_i\rangle\langle t_i| \otimes M_j]$, $q_k = \text{tr}[\rho_{AB}M_k \otimes |0\rangle\langle 0|]$, and $r_k = \text{tr}[\rho_{AB}M_k \otimes |1\rangle\langle 1|]$, where the states $\{|t_i\rangle\}_{i=1}^4$ are chosen, for convenience, from $\{|0\rangle, |1\rangle, (|0\rangle + |1\rangle)/\sqrt{2}, (|0\rangle + i|1\rangle)/\sqrt{2}\}$. From these probabilities, it is possible to reconstruct an operator after the SPA-PT in Eq. (1) and determine the eigenvalues using a determinant.

The above brute-force method is applied to demonstrate the entanglement detection of two-qubit states in the following form: $\rho(p, \alpha) = (1 - p)|\psi\rangle\langle\psi| + p|\psi^\perp\rangle\langle\psi^\perp|$ where $|\psi\rangle = \alpha|01\rangle - \sqrt{1 - |\alpha|^2}|10\rangle$. We generated nine different quantum states $\rho_k(p, \alpha)$ (for $k = 1, \ldots, 9$) and identified them using QST: $(p, \alpha) = [(0, 0.71),$ $(0.12, 0.71), (0.25, 0.71), (0.3, 0.71), (0.51, 0.71), (0, 0.92),$

$(0, 0.97), (0.37, 0.86), (0.42, 0.92)]$. Note that these (p, α) are obtained as the average of the data. In Fig. 3(a), these states $\{\rho_k\}_{k=1}^9$ are shown in terms of the linear entropy S_L (as a measure of mixedness) and the tangle τ as an entanglement measure [20].

For those quantum states $\{\rho_k\}_{k=1}^9$, we compare three cases of entanglement detection: a theoretical prediction, experimental results by realizing the SPA-PT, and the "brute-force" approach. Therefore, the minimum eigenvalue denoted by $\lambda_{\min}$ is obtained for each case as follows: $\lambda_{\min}^{\text{Th}}$ refers to the one from the ideal (theoretical) SPA-PT of the input state ρ_k, $\lambda_{\min}^{\text{Exp}}$ is computed from the QST of the resulting state after the SPA-PT in experiments, and $\lambda_{\min}^D$ is from the above-mentioned "brute-force" method. In Fig. 3(b), all these are compared in terms of the minimum eigenvalues and the tangle τ. We have also performed these for four Bell states and the results show that all minimum eigenvalues are indeed smaller than 2/9; see Table I. With these extensive examples, we have verified experimentally that a method for entanglement
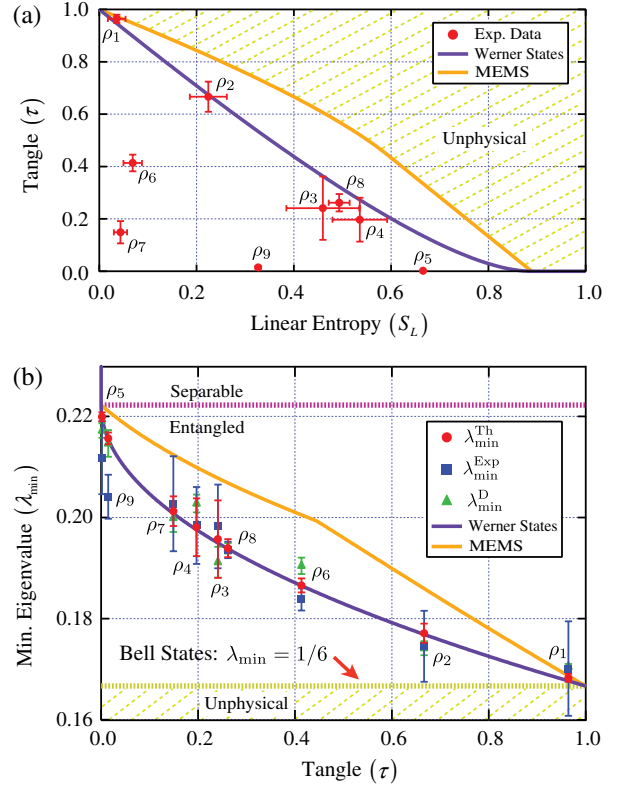

FIG. 3 (color online). (a) Nine quantum states $\{\rho_k\}_{k=1}^9$ are shown in the tangle (τ)—linear entropy (S_L) plane. (b) Minimum eigenvalues of resulting states after SPA-PT are shown. The error bars in (b) with respect to τ can be found in (a). Negative eigenvalues of two-qubit states after the PT is in the range $[-1/2, 0]$, which is $[1/6, 2/9]$ in the case of the SPA-PT. The Werner state $\rho_W = pI \otimes I/4 + (1 - p)|\psi^-\rangle\langle\psi^-|$ and the maximally entangled mixed states $\rho_{\text{MEMS}} = f(p)(|00\rangle\langle00| + |11\rangle\langle11|) + p(|00\rangle\langle11| + |11\rangle\langle00|)/2 + (1 - 2f(p))|01\rangle\langle01|$ where $f(p)$ is $p/2$ for $p \geq 2/3$ and $1/3$ for $p < 2/3$ [22].

TABLE I. Minimum eigenvalues $\lambda_{\min}$ obtained using three different methods described in the text for four Bell states shown in Fig. 2. In all cases, $\lambda_{\min}$ is much smaller than the threshold $2/9 \approx 0.222$ and, in fact, very close to the maximal entanglement case, $1/6 \approx 0.167$.

	$|\phi^+\rangle$	$|\phi^-\rangle$	$|\psi^+\rangle$	$|\psi^-\rangle$
$\lambda_{\min}^{\text{Th}}$	0.169	0.170	0.168	0.168
$\lambda_{\min}^{\text{Exp}}$	0.169	0.166	0.171	0.170
$\lambda_{\min}^{D}$	0.174	0.173	0.166	0.171

detection using the SPA-PT does largely depend on correlations existing in quantum states.

Conclusion.—We have proposed a practical scheme to realize the partial transpose operation via structural physical approximation and reported its experimental realization for two-qubit states in photonic systems using linear optics. We have also demonstrated entanglement detection using SPA-PT. The experimental results show that, contrasted to entanglement witnesses, the SPA-PT works with no dependence on local basis. Thus, with optimizations over measurement outcomes, SPA-PT demonstrated in this work would lead to direct, efficient, and practical methods of detecting entanglement, which is essential for any quantum information processing tasks. Furthermore, since our SPA-PT scheme is based on local measurements and classical communication, it can be applied to long-distance quantum information tasks [21]. In various contexts of quantum information processing, entanglement detection is generally and often a basic task required for quantum information applications. Our work therefore has immediate and wide ranging applications in quantum computation and quantum communication.

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