Percolation Transition in Yang-Mills Matter at a Finite Number of Colors

Stefano Lottini[1] and Giorgio Torrieri[2]

[1]ITP J. W. Goethe Universität, Max-von-Laue-Strasse 1, 60438 Frankfurt am Main, Germany
[2]FIAS, J. W. Goethe Universität, Max-von-Laue-Strasse 1, 60438 Frankfurt am Main, Germany
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We examine baryonic matter at a quark chemical potential of the order of the confinement scale $\mu_q \sim \Lambda_{\rm QCD}$. In this regime, quarks are supposed to be confined but baryons are close to the "tightly packed limit" where they nearly overlap in configuration space. We show that this system will exhibit a percolation phase transition when varied in the number of colors N_c: at high N_c, large distance correlations at the quark level are possible even if the quarks are essentially confined. At low N_c, this does not happen. We discuss the relevance of this for dense nuclear matter, and argue that our results suggest a new "phase transition," varying N_c at constant μ_q.

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Long ago, it was suggested to describe the deconfinement phase transition via percolation [1–7]. The idea is that, at increasing energies, the high parton density will make partons of different hadrons overlap. It is natural to associate this transition to deconfinement, where a quark can propagate throughout the hot medium rather than being confined to the hadron size $\sim\Lambda_{\rm QCD}^{-1} \sim 1$ fm in natural units [8]. Currently, we believe that the transition to quark-gluon plasma at low density is a crossover, casting doubt on the relevance of percolation (a phase transition, generally of second order) in those conditions.

In this work we use the percolation picture to study a different but related region in the phase diagram, the one at low to moderate temperature $0 \leq T \leq \Lambda_{\rm QCD}$ and high quark chemical potential $\mu_q \sim \Lambda_{\rm QCD}$. Strongly interacting matter in this regime has recently received a considerable amount of interest. Such matter can hopefully be produced in heavy-ion collisions [9–12], and is thought to exhibit a rich phenomenology, such as a critical point [13], spinodal instabilities [14], separation between chiral symmetry and confinement [15–18], chirally inhomogeneous phases [19–21], new phases [22], etc.

These conjectures are, however, extraordinarily difficult to explore quantitatively in a rigorous manner. The quark chemical potential μ_q is nowhere near the asymptotic freedom limit where perturbative QCD can be used [8]. It is, however, way too high for existing lattice-based approaches, dependent on $\mu_q/T \ll 1$, to work [23–25]. Hence, a simple geometric picture such as percolation might help. Its physical relevance is demonstrated in Fig. 1, schematically showing the regime where $\mu_q \sim \Lambda_{\rm QCD}$. At this chemical potential the density is naively expected to be $\sim\mathcal{O}(1)\Lambda_{\rm QCD}^3$, that is, one baryon per baryonic size. In configuration space, this means baryons touch each other; i.e., their quarks are separated by a scale not much larger than the confinement scale. We therefore expect some weakly coupled features of QCD be present due to the small interquark distance. Nonperturbative features, on the other hand, should also be present since baryons are still confined. The interplay of all these features could be very physically interesting. Such a setup was recently investigated in [22] using the only relevant quantity that can be called "a small parameter": $1/N_c$, with N_c number of colors [26]. The idea is to keep the quantity $\lambda = g^2 N_c$ [where g is the $SU(N)$ Yang-Mills coupling constant [8]] fixed, sending the number of colors N_c to infinity, and then expanding in $1/N_c$. It is easy to see that the running of λ is qualitatively N_c independent, hence $\Lambda_{\rm QCD} \sim N_c^0$. Obviously, such a setup can give at best a qualitative agreement at $N_c = 3$, but it might be enough to understand the phase diagram structure of the system. In this setup each quark-quark interaction is weak ($\sim \lambda/N_c$) but, due to combinatorics, baryons remain strongly coupled "semiclassical" objects [27]. Looking at Fig. 1 with a large-N_c perspective, one can immediately see that quarks may be arbitrarily close together in configuration space (interquark distance $\sim N_c^{-1/3}$), so interactions of quarks between neighboring baryons could be weakly

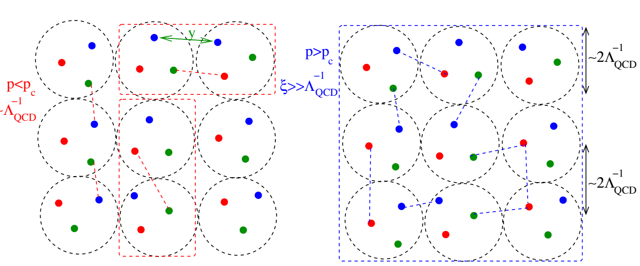

FIG. 1 (color online). The structure, in configuration space, of dense baryonic matter. Comparison of the top and the bottom panels shows that the percolation picture is applicable: when the exchange probability goes above the threshold p_c the size of the typical cluster diverges. The figure also shows a definition of the interquark distance y of Eq. (3).

coupled. Confinement, however, persists until quark-hole screening $\sim$ gluon antiscreening [22], $\mu_q \sim N_c^{1/2}\Lambda_{\rm QCD}$.

How can a confined system be, at the same time, asymptotically free? The authors of [22] conjectured that excitations below the Fermi surface behave as asymptotically free quarks, but excitations close to the Fermi surface are confined. Thus, the entropy of such a system is $\sim N_c$ (unlike N_c^0 for confined matter), but dynamical excitations are baryons and mesons and the Polyakov loop expectation value [8] is ~ 0. This new "quarkyonic" state of matter [22] might be realized in our $N_c = 3$ world.

The problem is that it is difficult to model this phase theoretically in a rigorous way. While quarkyonic matter has been claimed to be found in the Polyakov-Nambu-Jona-Lasinio (pNJL) model [18], one cannot reliably investigate whether this phase has pressure $\sim N_c$ in that model, since the pressure in pNJL is always somewhat $\sim N_c$, since the "Polyakov loop field" as implemented in [15,16,18] is different from "true" confinement. Similarly, it is difficult to see how holographic methods [28] can verify such a conjecture, since there λN_c^{-1} plays the role of the string coupling constant g_s (λ is the compactification scale of the remaining 5 dimensions in units of Planck length); in a semiclassical gravity regime, such as [29], pressure in a confined phase is always $\sim N_c^0$ [30,31]. Taking these difficulties into account, it is not clear how applicable is the large N_c limit to our $N_c = 3$ world. Nuclear matter seems to look very different in that world compared to ours, as it is a tightly bound crystal of baryons [27,32,33] (unless what we call large-N_c matter is actually the quarkyonic phase). Mean field analysis show a considerable N_c variation between $N_c = 3$ and $10 > N_c > 3$ [34]. In [35] a physical interpretation of these structures was conjectured: due to the Pauli exclusion principle, an interplay exists between the number of colors N_c and the number of neighbors N_N. Hence, a "phase transition" exists in N_c space when the baryon density is $\sim\Lambda_{\rm QCD}^3$, separating our world ($N_c \ll N_N$) from the truly large-N_c world. In this Letter, we aim to apply standard percolation knowledge to investigate this further.

The key insight suggesting that interesting structures might be lurking in N_c is that 3D bond percolation exhibits a phase transition at comparatively low critical link probability: for instance, $p_c \sim 0.25$, 0.18, 0.12 for simple-cubic (schematically shown in Fig. 1), body-centered-cubic, and hexagonal-close-packed lattice, respectively [36]. Such values suggest that long-distance correlations on the quark level could occur even with a somewhat low percentage of quarks hopping between baryons, i.e., firmly in the confined phase. While below p_c the characteristic correlation distance ξ ($\sim$ cluster size) is $\sim\Lambda_{\rm QCD}^{-1}$, above the threshold this quantity explodes to the total system size in a comparable amount of time. We leave the meaning of "correlation" vague, as it can be either a quark hop or a gluon exchange; in our context, it implies exchange of color degrees of freedom within a confined tightly packed medium. We encode the likelihood of exchange between neighboring baryons in a link probability p, to be compared with the percolation threshold p_c in order to assess the formation of large-scale structures.

Two baryons will be correlated if at least two quarks are correlated. One has to sum over all possible multiquark configurations, resulting in a strong N_c dependence of p. We determine the latter by calculating the probability $q = 1 - p$ that no exchanges happen between neighboring baryons. We define $p_{(1),ij}$ as the probability that quarks i and j, respectively, in baryons A and B "correlate" (either by flip or gluon exchange). Assuming the quarks inside the nucleon are uncorrelated (Fermi motion dominates), this probability factorizes into a geometric distribution $f_{A,B}(\mathbf{x})$ for quarks to be at a certain (vector) position $\mathbf{x}$ (see Fig. 1), and a "squared propagator" transition amplitude $F(d)$ for them:

$$p = 1 - (q_{(1),ij})^{(N_c)^\alpha};$$
$$q_{(1),ij} = \int f_A(\mathbf{x}_i)d\mathbf{x}_i \int f_B(\mathbf{x}_j)d\mathbf{x}_j(1 - F(|\mathbf{x}_i - \mathbf{x}_j|)). \quad (1)$$

We assume a "hard-sphere" distribution for $f_{A,B}$ (since we keep μ_q fixed, the distance between centers of neighboring baryons is always $2\Lambda_{\rm QCD}^{-1}$, Fig. 1):

$$f_{A,B}(\mathbf{x}) \propto \Theta(1 - \Lambda_{\rm QCD}|\mathbf{x} - \mathbf{x}_{A,B}^{\rm center}|) \quad (2)$$

and a probability of exchange $i \leftrightarrow j$ based on a range of "reasonable" propagators, compatible with confinement (fast fallout in configuration space at distances greater than $r_T \sim 1$ in units of $\Lambda_{\rm QCD}^{-1}$) and with the large N_c limit of QCD, the interaction is $\sim g^2 \sim \lambda/N_c$ [26,27]. The propagators we use are the simple Θ function in configuration space and the momentum-space Θ function used in [19], all normalized so their area is $\lambda r_T/N_c$. In configuration space the transition amplitudes are, respectively,

$$F(y) = \frac{\lambda}{N_c}\begin{cases} \Theta\left(1 - \frac{y\Lambda_{\rm QCD}}{r_T}\right), \\ \frac{2r_T^2}{\pi y^2}\sin^2\left(\frac{y\Lambda_{\rm QCD}}{r_T}\right). \end{cases} \quad (3)$$

Other transition amplitudes, such as a Gaussian distribution in configuration space, were also tried with no significant modifications of the results presented below.

We note that, due to the fact that 3D percolation has a second-order phase transition at a certain $p_c < 1$, the results we obtain below have some degree of universality: as long as the qualitative features of confinement are observed [the transition amplitude $F(y)$ drops sharply above the scale r_T, and the hadron density profile $f_{A,B}(\mathbf{x})$ has a central plateau of radius $\sim\Lambda_{\rm QCD}^{-1}$ and a sharply decreasing tail outside], the results we show vary quantitatively but not qualitatively, in particular, in regard to phase transition behavior if $p \sim p_c$ for some critical N_c. Mathematically, this model is similar to the Glauber model, familiar in heavy-ion collisions [8], with the number of colors playing the role of the number of participants $N_{\rm part}$. Just as in the Glauber model the dependence on $N_{\rm part}$ is universal with respect to cross sections, in this case the dependence on N_c is universal with respect to the shape of the transition amplitude.

The crucial parameter left is α in Eq. (1). Here, we shall consider two limits: one can assume that the link of quark i from baryon A to quark j of baryon B does not prejudice in any way the possibility of also linking i-k, with quark k again in baryon B. This scenario, natural if the link is actually realized by a gluon exchange rather than a quark flip, means that $\alpha = 2$. If, on the other hand, the link is given by a quark exchange, then, on a short enough time step, the probability of a quark moving more than once is negligible. In this case, $\alpha = 1$.

Physically one expects that at $N_c \gg 1$ gluon exchange dominates over quark flip, by combinatorics alone. Indeed, one can easily see that $\alpha = 1$ is in contradiction with the Skyrme crystal picture at large N_c: in this picture, $p(N_c)$ approaches a constant large-N_c value from above: low N_c nuclear matter would be more correlated (and hence more strongly bound) than high N_c nuclear matter. Comparing strongly coupled $N_c \to \infty$ nuclear matter [27] to the weakly bound nuclear liquid at $N_c = 3$ [35], this is obviously not right. We therefore assume $\alpha = 2$ henceforward. A closer inspection of the $\alpha = 2$ case reveals that it is an approximation of

$$p = 1 - \int \prod_{\ell=1}^{N_c} f_A(\mathbf{x}_\ell^{(A)}) d\mathbf{x}_\ell^{(A)} \times \int f_B(\mathbf{x}_m^{(B)}) \prod_{m1}^{N_c} d\mathbf{x}_m^{(B)} \prod_{i,j} (1 - F(|\mathbf{x}_i^{(A)} - \mathbf{x}_j^{(B)}|)), \quad (4)$$

where we integrate over all quark positions in the two baryons and write the probability of no exchanges taking place as the product of the individual no-exchanges probabilities for N_c^2 A-B pairs. Numerical integration shows the effect of correlations to be a $\sim 3\%$ correction, so that the qualitative outcome of the analysis is unaffected.

In the large N_c limit for the case $\alpha = 2$, p asymptotically approaches unity. It is reasonable that this is the point where the "dense baryonic matter as a Skyrme crystal," theorized in [22,27,32,33], is reached. If this is the case, however, one should remember that a percolation second-order phase transition occurs at a $p_c \ll 1$. Hence, keeping $\mu_q \sim \Lambda_{\rm QCD}$ fixed but varying N_c, the features of the Skyrme crystal should manifest not with a continuous approach, rather as a second-order transition at a not too high N_c, whose order parameter can be thought to be the "giant cluster" density. Below the critical N_c there is little correlation between quarks of different baryons, while above this threshold they can correlate, with the distance boundary given only by causality. We reiterate that this is not deconfinement since $\mu \sim \Lambda_{\rm QCD} \ll N_c^{1/2} \Lambda_{\rm QCD}$ independently of the number of colors, and the fraction of correlated quarks from different hadrons is still $\sim$0.1–0.3 $\ll 1$ at the percolation transition. Right above this transition, therefore, the baryonic wave function should not be too different from the large-N_c baryonic wave function described in [27]. The correlation distance of quarks will, however, be much larger than the baryon size. The features of this new phase are therefore similar to those of the quarkyonic matter [22].

Assuming the lowest 3D value $p_c = 0.12$, appropriate for a closely packed hexagonal lattice, the critical number of colors is shown in Fig. 2 as a function of λ and r_T. As can be seen, the critical number of colors is significantly larger than 3 for $r_T \sim \Lambda_{\rm QCD}^{-1}$, $\lambda \sim 1$. However, given the roughness of our model, a critical $N_c \leq 3$ cannot be excluded at $\mu_q \sim \Lambda_{\rm QCD}$, provided quarks can correlate

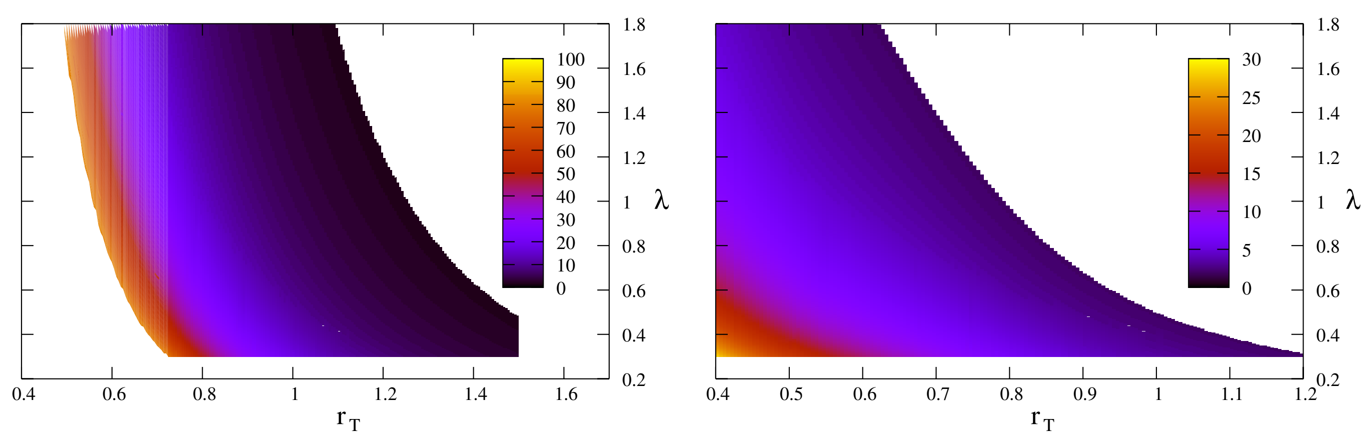

FIG. 2 (color online). Contour plot of the critical N_c for the percolation transition in a hexagonally packed lattice as a function of the coupling λ and range r_T (in $\Lambda_{\rm QCD}$ units). The left panel assumes a Θ-function correlation probability in configuration space, the right panel assumes a correlation probability based on the propagator used in [19]. Diagram covers $2 \leq N_c \leq 80$.

significantly above the confinement length ($\sim 1.5\Lambda_{\rm QCD}^{-1}$) or the coupling constant is significantly larger than unity (one has to remember, however, that these quantities are not independent: $\Lambda_{\rm QCD}$ is defined as the scale at which λ becomes "strong," ~ 1, and it is generally assumed that confinement is set by that scale).

Considering Fig. 2 is a lower limit since p_c is at its minimum (p_c is significantly higher both in a Skyrme cubic crystal and in a disordered fluid), we can say that $N_c = 3$ is disfavored, although it cannot be excluded. Changing temperature and μ_q should further change the critical N_c. Exploring this parameter space, and seeing how it relates to the confinement phase transition, is the subject of a forthcoming work.

What are the phenomenological consequences of percolation? If by correlation we mean energy-momentum-exchange via quark tunneling between baryons, it is reasonable that pressure and entropy density $\sim N_c$ above the percolation threshold, while below it they stay $\sim N_c^0$. This is because above the threshold, where interbaryon tunneling is significant, "typical" excitations of the Fermi surface will be superpositions across baryons of baryon-localized quark-hole excitations, similar to conduction band electrons in a metal; while the localized excitation energy $\geq \Lambda_{\rm QCD}$, the superposition makes its energy $\sim \Lambda_{\rm QCD}$ even if color degeneracy remains. Thus, the degrees of freedom of the system above percolation will be delocalized weakly interacting quarks in a lattice of confining potentials, a picture compatible with [20,22]. Below the threshold, where tunneling is negligible, excitations are either color singlets or of energy $E \gg \Lambda_{\rm QCD}$, suppressed below deconfinement.

The only known rigorous way to access these phenomena quantitatively is the lattice. Current quenched simulations show that N_c dependence is surprisingly smooth [37,38]. Our considerations suggest that this will not be true at finite chemical potential. While chemical potentials accessible to current numerical studies are far smaller than dense-packing densities [23–25], approaches such as the strong coupling expansion [39] could be used to probe the large-N_c dependence at these densities.

This transition might also be visible with holographic techniques beyond the supergravity limit, since finite λN_c^{-1} corresponds, in gauge-string duality, to g_s [28]. Thus, percolation will manifest itself as a transition from "stringy weak coupling" to "not-so-weak coupling." While there are hints [40,41] that percolation is relevant for corrections beyond classical gravity, exploring this is beyond the scope of this work. The fact that percolation appears only as a subleading factor of g_s might explain why, despite the reasonableness of the argument in [22] for $s \sim N_c$ in the quarkyonic phase, $s \sim N_c^0$ in the confined phase in all semiclassical AdS/CFT setups to date.

In conclusion, we have used a toy model, with universal features, to investigate the close-packed regime ($\mu_q \sim \Lambda_{\rm QCD}$) of baryons at variable N_c. Our findings suggest that, if baryons are kept in this regime but N_c is varied, a percolation-type phase transition occurs at some critical $N_c \sim \mathcal{O}(10)$, probably but not certainly higher than 3. This transition is quite distinct from deconfinement as the percentage of quarks propagating on superbaryonic distances is quite low. Nevertheless, the typical correlation length will be much larger than $\Lambda_{\rm QCD}^{-1}$. Further work needs to be done to explore the phenomenological consequences of this transition, but our findings suggest that applying the $N_c \to \infty$ limit to dense baryonic systems should be done with caution, since a discontinuity might be present between $N_c = 3$ and $N_c \to \infty$.

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