Collective Uncertainty Entanglement Test

Łukasz Rudnicki,[1,*] Paweł Horodecki,[2,3] and Karol Życzkowski[1,4]

[1]Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/46, PL-02-668 Warsaw, Poland
[2]Faculty of Applied Physics and Mathematics, Technical University of Gdańsk, PL-80-952 Gdańsk, Poland
[3]National Quantum Information Centre of Gdańsk, PL-81-824 Sopot, Poland
[4]Smoluchowski Institute of Physics, Jagiellonian University, ul. Reymonta 4, PL-30-059 Kraków, Poland

(Received 17 June 2011; published 3 October 2011)

For a given pure state of a composite quantum system we analyze the product of its projections onto a set of locally orthogonal separable pure states. We derive a bound for this product analogous to the entropic uncertainty relations. For bipartite systems the bound is saturated for maximally entangled states and it allows us to construct a family of entanglement measures, we shall call collectibility. As these quantities are experimentally accessible, the approach advocated contributes to the task of experimental quantification of quantum entanglement, while for a three-qubit system it is capable to identify the genuine three-party entanglement.

DOI: 10.1103/PhysRevLett.107.150502 PACS numbers: 03.67.Mn, 03.65.Ud

The phenomenon of quantum entanglement—nonclassical correlations between individual subsystems—is a subject of intense research interest [1–3]. Several criteria of detecting entanglement are known [2,3], and some of them can be implemented experimentally (see [4] for the review of specific experimental schemes). In particular, the issue of qualitative entanglement detection is quite well established including the entanglement witnesses method (see [3]) and local uncertainty relations [5]. On the other hand, although various measures of quantum entanglement are analyzed [3,6], in general they are more difficult to quantitatively measure in a physical experiment. To estimate experimentally the degree of entanglement of a given quantum state, one usually relies [7] on quantum tomography or analogous techniques.

The idea of entanglement detection and estimation without prior tomography [8,9] involves the collective measurement of two (or more) copies of the state as demonstrated in [10]. Consequently, recent attempts towards experimental quantification of entanglement are based on finding collectively measurable quantities which bound known entanglement measures from below and are experimentally accessible [11,12] (for review, see [13]).

The main aim of this work is to construct a family of indicators, designed to quantify the entanglement of a pure state of an arbitrary composite system, which can be measured in a coincidence experiment without attempting a complete reconstruction of the quantum state.

Our approach, which leads to a simple collective entanglement test, is inspired by the entropic uncertainty relations which are satisfied by any pure state. For instance, the sum of the Shannon entropies of the expansion coefficients of a given pure state $|\psi\rangle \in \mathcal{H}_N$ expanded in two mutually unbiased bases is bounded from below by $\ln N$ [14]. This observation suggests to quantify the pure states' entanglement by a function of the projections of the analyzed state $|\Psi\rangle$ of a composite system onto mutually orthogonal separable pure states.

The method we propose can be formulated in a rather general case of a normalized pure state, $\langle\Psi|\Psi\rangle = 1$, of a composite system consisting of K subsystems. For simplicity we shall assume here that all their dimensions are equal, so we consider an element of a K-partite Hilbert space $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \otimes \cdots \otimes \mathcal{H}^K$, where $\dim(\mathcal{H}^A) = \cdots = \dim(\mathcal{H}^K) = N$. Let us select a set of N separable pure states of a K-qudit system, $|\chi_j^{\text{sep}}\rangle = |a_j^A\rangle \otimes \cdots \otimes |a_j^K\rangle$, where $|a_j^I\rangle \in \mathcal{H}^I$ with $j = 1, \ldots, N$ and $I = A, \ldots, K$. The key assumption is that all local states are mutually orthogonal, so that

$$|a_1^I\rangle, \ldots, |a_N^I\rangle \in \mathcal{H}^I, \qquad \langle a_j^I|a_k^I\rangle = \delta_{jk}. \tag{1}$$

Entanglement detection.—In order to construct measurable indicators of quantum entanglement and find practical entanglement criteria valid for any analyzed state $|\Psi\rangle$, we define now the following quantity:

$$Y^{\max}[|\Psi\rangle] = \max_{|\chi^{\text{sep}}\rangle} \prod_{j=1}^{N} |\langle\Psi|\chi_j^{\text{sep}}\rangle|^2. \tag{2}$$

This product of the projections of the state onto the set of N separable states, optimized over all possible sets of mutually locally orthogonal states, $|\chi^{\text{sep}}\rangle = \{|\chi_1^{\text{sep}}\rangle, \ldots, |\chi_N^{\text{sep}}\rangle\}$, will be called maximal *collectibility*.

Note the difference with respect to the geometric measure of entanglement [15], to define which one takes the maximum over a single separable state, $|\chi_1^{\text{sep}}\rangle$. In this case this maximum, denoted in [15] by $\Lambda_{\max}^2$, is equal to unity if the analyzed state $|\Psi\rangle$ is separable and it is smaller for any entangled state, so to define the geometric measure of entanglement one takes $1 - \Lambda_{\max}^2$. In contrast, taking in (2) the maximum of the product of the projections of $|\Psi\rangle$ onto $N \geq 2$ separable states $|\chi_j^{\text{sep}}\rangle$, we face an inverse situation: we show below that $Y^{\max}$ is the largest for maximally entangled states, so this quantity can serve directly as a quantificator of entanglement.

To this end we shall start with a variational equation

$$\frac{\delta}{\delta|\Psi\rangle}\left(\prod_{j=1}^{N}|\langle\Psi|\chi_j^{\rm sep}\rangle|^2-\lambda\langle\Psi|\Psi\rangle\right)=0, \qquad (3)$$

where λ plays the role of a Lagrange multiplier associated with the normalization constraint. This idea was developed by Deutsch in order to obtain the entropic uncertainty relation [16]. Equation (3) implies

$$\prod_{j=1}^{N}|\langle\Psi|\chi_j^{\rm sep}\rangle|^2\sum_{i=1}^{N}(\langle\chi_i^{\rm sep}|\Psi\rangle)^{-1}\langle\chi_i^{\rm sep}|=\lambda\langle\Psi|. \qquad (4)$$

Multiplying (4) by $|\Psi\rangle$ we find out that $\lambda=N\prod_{j=1}^{N}|\langle\Psi|\chi_j^{\rm sep}\rangle|^2$. Moreover, the contraction of (4) with $|\chi_m^{\rm sep}\rangle$ leads to $|\langle\Psi|\chi_m^{\rm sep}\rangle|^2=1/N$ for all values of m. From this result we have

$$\max_{|\Psi\rangle}\prod_{j=1}^{N}|\langle\Psi|\chi_j^{\rm sep}\rangle|^2=\prod_{j=1}^{N}\frac{1}{N}=N^{-N}, \qquad (5)$$

which after formal optimization over $|\chi^{\rm sep}\rangle$ implies the desired inequality

$$Y^{\max}[|\Psi\rangle]\leq N^{-N}. \qquad (6)$$

Using an auxiliary variable, $Z^{\max}=-\ln Y^{\max}$, this relation takes the from $Z^{\max}[\Psi]\geq N\ln N$, analogous to the entropic uncertainty relation. Interestingly, for a bipartite system this inequality is saturated for the maximally entangled state, $|\Psi_+\rangle=\frac{1}{\sqrt{N}}\sum_i|i,i\rangle$, while in the case of the K-qudit system it is saturated for a generalized Greenberger-Horne-Zeilinger (GHZ) state, $|{\rm GHZ}\rangle_K=\frac{1}{\sqrt{N}}\sum_i|i\rangle_A\otimes\cdots\otimes|i\rangle_K$.

Consider now the other limiting case of a separable state $|\Psi_{\rm sep}\rangle=|\Psi_A\rangle\otimes\cdots\otimes|\Psi_K\rangle$. In this case the projections factorize,

$$\langle\Psi_{\rm sep}|\chi_j^{\rm sep}\rangle=\prod_{I=A}^{K}\langle\Psi_I|a_j^I\rangle. \qquad (7)$$

Furthermore, for each value of the index ($I=A,\ldots,K$) we can independently apply the result (5) and obtain

$$\prod_{j=1}^{N}|\langle\Psi_I|a_j^I\rangle|^2\leq N^{-N}. \qquad (8)$$

Thus, for any separable state we have

$$\prod_{j=1}^{N}|\langle\Psi_{\rm sep}|\chi_j^{\rm sep}\rangle|^2\leq\prod_{I=A}^{K}\max_{|\Psi_I\rangle}\prod_{j=1}^{N}|\langle\Psi_I|a_j^I\rangle|^2=N^{-NK}, \qquad (9)$$

so that

$$Y^{\max}[|\Psi_{\rm sep}\rangle]\leq N^{-NK}. \qquad (10)$$

This observation leads to the following separability criteria based on the maximal collectibility:

$$(Y^{\max}[|\Psi\rangle]>\alpha_{K,N})\Rightarrow(|\Psi\rangle-\text{entangled}). \qquad (11)$$

Here $\alpha_{K,N}=N^{-NK}$ is the discrimination parameter.

Multiqubit systems.—In the definition (2) of the maximal collectibility, one performs a maximization over the set of all N mutually orthogonal separable states $|\chi_j^{\rm sep}\rangle$. The maximal collectibility $Y^{\max}$ can be considered as a pure state entanglement measure, and we derive below its explicit expression in the simplest case of a two-qubit system. However, it is also convenient to perform the optimization procedure stepwise and to consider first an optimization over a single Hilbert subspace.

Let us then define a one-step maximum over the local states belonging to the first subspace $\mathcal{H}^A$,

$$Y_a[|\Psi\rangle]=\max_{|a^A\rangle}\prod_{j=1}^{N}|\langle\Psi|\chi_j^{\rm sep}\rangle|^2. \qquad (12)$$

Note that the collectibility Y_a, a function of the analyzed state $|\Psi\rangle$, is parametrized by the set a of N product states $|a_j^B\rangle\otimes\cdots\otimes|a_j^K\rangle$, with $j=1,\ldots,N$. By construction one has $\max_a Y_a[|\Psi\rangle]=Y^{\max}[|\Psi\rangle]$.

Consider now the case of a K-qubit system ($N=2$). Writing an equation analogous to (3) and following the standard variational approach, we obtain an analytical formula for the collectibility,

$$Y_a[|\Psi\rangle]=\frac{1}{4}(\sqrt{G_{11}G_{22}}+\sqrt{G_{11}G_{22}-|G_{12}|^2})^2, \qquad (13)$$

expressed in terms of elements of the Gram matrix defined for a set of projected states. Here $G_{jk}=\langle\varphi_j|\varphi_k\rangle$, while $|\varphi_j\rangle\in\mathcal{H}^A$ denotes the state $|\Psi\rangle$ projected onto the jth separable state living in $K-1$ subspaces labeled by $B,\ldots,K$, so that $|\varphi_j\rangle=[\langle a_j^B|\otimes\cdots\otimes\langle a_j^K|]|\Psi\rangle$.

Because of (5) and (9) the collectibility Y_a satisfies the same uncertainty relations (6) and (10) as the maximal collectibility $Y^{\max}$. This approach can be generalized to the case of Hilbert spaces with different dimensions. It can be especially useful when $\dim(\mathcal{H}^A)$ is much larger than the dimensions of remaining Hilbert spaces. This case may for instance describe the entanglement with an environment.

Two qubits.—Let us now investigate in more detail the simplest case of a two-qubit system for which $K=N=2$ and $\mathcal{H}=\mathcal{H}^A\otimes\mathcal{H}^B$. Any pure state $|\Psi\rangle_{AB}$ can then be written in its Schmidt form [2],

$$|\Psi\rangle_{AB}=(U_A\otimes U_B)\left[\cos\left(\frac{\psi}{2}\right)|00\rangle+\sin\left(\frac{\psi}{2}\right)|11\rangle\right], \qquad (14)$$

where $U_A\otimes U_B$ is a local unitary. The Schmidt angle $\psi\in[0,\pi]$ is equal to zero for the separable state and to $\pi/2$ for the maximally entangled state. From the uncertainty relation (6) we know that $Y_a[|\Psi\rangle_{AB}]\leq 1/4$. Moreover, if the state (14) is separable, we have (10) $Y_a[|\Psi_{\rm sep}\rangle]\leq 1/16$.

Now we assume the general form of the orthonormal detector basis spanned in the second subspace $\mathcal{H}^B$,

$$|a_1^B\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\imath\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle,$$
$$|a_2^B\rangle = \sin\left(\frac{\theta}{2}\right)|0\rangle - e^{\imath\phi}\cos\left(\frac{\theta}{2}\right)|1\rangle, \qquad (15)$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. Because of this general form, our analysis becomes independent of the local unitary U_B in (14). Note also that the expression (13) is independent of U_A; thus, our approach works universally for any two-qubit pure state. Using (15) we shall calculate the entries of the Gram matrix and find

$$Y_\theta(\psi) = \frac{[2\sin(\psi) + \sqrt{3 - 2\cos(2\theta)\cos^2(\psi) - \cos(2\psi)}]^2}{64}.$$

The collectibility $Y_\theta(\psi)$ depends on the analyzed state (ψ) and the detector parameters $a = (\theta, \phi)$. The dependence on the azimuthal angle ϕ is trivial. If the state (14) is maximally entangled ($\psi = \pi/2$) then $Y_\theta(\pi/2) = 1/4$ and the collectibility attains its maximal possible value independently of the choice of (θ, ϕ).

In order to characterize various possibilities to detect the entanglement, we analyze four quantities. Consider first the minimal $Y^{\min} = \min_\theta Y_\theta = Y_0$ and the maximal $Y^{\max} = \max_\theta Y_\theta = Y_{\pi/2}$ values of the collectibility $Y_\theta(\psi)$ with respect to the detector parameters (θ, ϕ),

$$Y^{\min}(\psi) = \frac{\sin^2(\psi)}{4}, \qquad Y^{\max}(\psi) = \frac{[1 + \sin(\psi)]^2}{16}.$$

Then define the mean collectibility $\bar{Y} = \langle Y_a\rangle_a$, averaged over the set of the detector parameters $a = (\theta, \phi)$ with the measure $d\Omega = \sin(\theta)d\theta d\phi/(4\pi)$. This case, corresponding to the average over a random choice of the detector parameters, $\bar{Y}(\psi) = \int_{\mathcal{S}^2} d\Omega Y_\theta(\psi)$, yields the result

$$\bar{Y}(\psi) = \frac{11 - 7\cos(2\psi) + 3(\pi - 2\psi)\tan(\psi)}{96}. \qquad (16)$$

Furthermore, we study the probability that the entanglement is detected in a measurement with a random choice of the detector angle θ, $P_Y(\psi) = \int_{\mathcal{P}} d\Omega$, where $\mathcal{P} = \{(\theta, \phi) \in \mathcal{S}^2 : Y_\theta(\psi) > 1/16\}$:

$$P_Y(\psi) = \begin{cases} \frac{\sqrt{2\sin(\psi) - \sin^2(\psi)}}{|\cos(\psi)|} & \text{for } \psi \in [0, \frac{\pi}{6}] \cup [\frac{5\pi}{6}, \pi] \\ 1 & \text{for } \psi \in [\frac{\pi}{6}, \frac{5\pi}{6}]. \end{cases} \qquad (17)$$

Analytical results for a pure state of the 2×2 system are presented in Fig. 1. In the case of the optimal choice of the detector parameters (solid red curve) the entanglement is detected for any entangled state. More importantly, in the case of the worst possible choice of the measurement parameters represented by the dotted blue curve, the entanglement is detected for $\psi \in [\pi/6, 5\pi/6]$. This coincides with the fact that the probability of entanglement detection with a single random measurement is equal to unity [cf. (18) and the dashed black curve]. The average collectibility $\bar{Y}$ corresponds to an average obtained by a sequence of measurements with a random choice of the detector parameters. Looking at the expression (13), we see that to compute the collectibility Y_a it is enough to determine the elements of the Gram matrix. Assume first that we analyze a two-photon polarization-entangled state. The diagonal element G_{jj} represents an amplitude of the state $|\varphi_j\rangle$ in the first subspace $\mathcal{H}^A$, under the assumption that the second photon was measured by the detector in the state $|a_j^B\rangle$. To determine the absolute value of the off-diagonal element, $|G_{12}|^2 = |\langle\varphi_1|\varphi_2\rangle|^2$, of the two-photon state $|\Psi\rangle_{AB}$, one projects the $\mathcal{H}^B$ part of the first copy onto the state $|a_1^B\rangle$, the same part of the second copy onto $|a_2^B\rangle$, and performs a kind of the Hong-Ou-Mandel interference experiment [17] with the remaining two photons of the first subsystem $\mathcal{H}^A$. A specific scheme of this kind is depicted in Fig. 2.

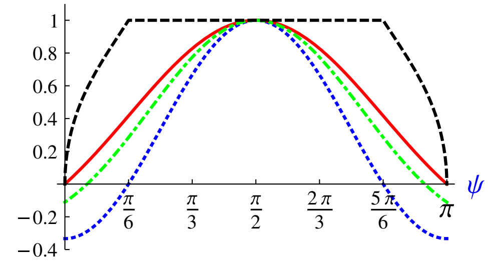

FIG. 1 (color online). Parameters describing entanglement of a two-qubit pure state $|\Psi\rangle_{AB}$ as a function of the Schmidt angle ψ. We plot the minimal (dotted blue curve), the average (dash-dotted green curve), and the maximal (solid red curve) values of the rescaled collectibility $[16Y_\theta(\psi) - 1]/3$. Positive values identify entanglement. The dashed black line shows the probability P_Y that the entanglement of $|\Psi\rangle_{AB}$ is detected in a particular random measurement.

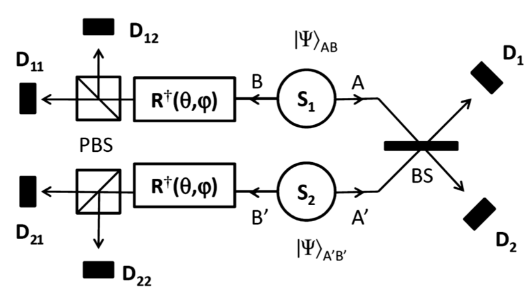

FIG. 2. Determination of the Gram matrix via conditional overlapping in the case of two polarization-entangled photon pairs. Each source produces a pair of photons in a polarization state $|\Psi\rangle_{AB}$. On the left-hand side B the statistics of pairs of clicks after two PBS elements are measured. On the right-hand side A the Hong-Ou-Mandel interference is performed. The number $|G_{12}|^2$ is equal to the probability of the pair of the clicks at B multiplied by that of double click at A.

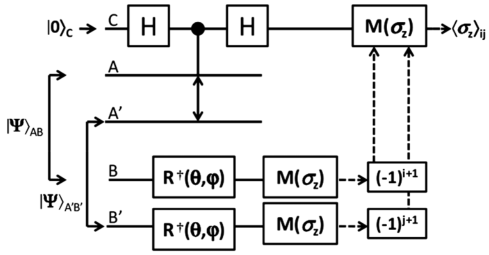

FIG. 3. Quantum network exploiting two copies of an analyzed state $|\Psi\rangle_{AB}$, a control qubit $|c\rangle$ initially in state $|0\rangle$, controlled SWAP gate (cf. [8]), and two Hadamard gates. The mean value of Pauli σ_z matrix of $|c\rangle$ is measured under the condition that the chosen pair (i, j) of results is obtained in measurement of the same observables performed on both qubits at the bottom of the scheme.

Apart from two sources of pure entanglement (which may base on type-I parametric down-conversion sources modified by dumping one of the polarization components), it involves the 50:50 beam splitter (BS), two polarization rotators $R^\dagger(\theta, \phi)$ in the same setting, and the polarized beam splitters (PBS). If by $p_{ij}(+,+)$ we denote the probability of double click after the beam splitter, and by $p_{1i} \equiv p_1((-1)^{i+1})$ $[p_{2i} \equiv p_2((-1)^{i+1})]$ the probability of click in the $D_{1,i}$th detector ($D_{2,i}$th detector), i.e., one of the detectors located after upper PBS (lower PBS), then all the Gram matrix elements are

$$|G_{ij}|^2 = p_{1i}p_{2j}(1 - 2p_{ij}(+,+)). \tag{18}$$

Alternatively one can apply the following network designed to measure all three quantities (see Fig. 3). Measuring the σ_z component of the first qubit, conditioned by a pair of the results (i, j) [coming with probabilities $p_1((-1)^{i+1})$, $p_2((-1)^{j+1})$] of the measurements of the same (σ_z) observable on the last two qubits one gets an estimation of the parameter $|G_{ij}|^2 = p_1((-1)^{i+1})p_2((-1)^{j+1})\langle\sigma_z\rangle_{ij}$.

Without going into detailed analysis here we only mention that the purity assumption may be dropped at the price of performing two variants of the experiment each with one of two complementary (in Heisenberg sense) settings $R^\dagger(\theta, \phi), R^\dagger(\theta', \phi')$. Then the discrimination parameter $\alpha_{K,N} = \alpha_{2,2} = 1/16$ in the inequality (11) may be successfully corrected by the term involving the impurities (measured by Hong-Ou-Mandel interference) of the states generated by measurements of the two observables [18].

Three qubits.—Now let us investigate the case of a three-qubit state ($K = 3$). In this case the separability discrimination parameter is equal to $\alpha_{3,2} = 1/64$. We compare a biseparable state $|BS\rangle = |\Psi\rangle_{AB} \otimes |\phi\rangle_C$ and two of the most important representatives, the GHZ state and the W state,

TABLE I. Comparison between the GHZ state, the W state, and the biseparable state $|BS\rangle$. We present numerical values for the minimal, maximal, and average collectibilities and the probabilities P_Y of entanglement detection in a particular, random measurement. If the values of the collectibility are larger than $1/64 \approx 0.016$, then the entanglement is detected.

Entanglement test	GHZ state	W state	BS state
Minimal $Y^{\min}$	0	0	0
Maximal $Y^{\max}$	0.250	0.141	0.063
Average $\bar{Y}$	0.053	0.049	0.021
Detection probability P_Y	0.807	0.807	0.500

$$|\mathrm{GHZ}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}},$$
$$|W\rangle = \frac{|001\rangle + |010\rangle + |100\rangle}{\sqrt{3}}.$$

Numerical results for the collectibility are compared in Table I. We can see that the maximal and average collectibilities detect entanglement of all three states. The maximum value is attained for the GHZ state, $Y^{\max}[|\mathrm{GHZ}\rangle] = 16/64$ while $Y^{\max}[|W\rangle] = 9/64$. As this quantity for the biseparable state reads $Y^{\max}[|BS\rangle] = 4/64$ and $Y^{\max}[|\Psi_{\mathrm{sep}}\rangle] = 1/64$, the collectibility offers an experimentally accessible measure capable to distinguish the genuine three-parties entanglement.

It is a pleasure to thank K. Banaszek, O. Gühne, M. Kuś, and M. Żukowski for fruitful discussions and helpful remarks. Financial support by Grants No. N N202 174039, No. N N202 090239, and No. N N202 261938 of the Polish Ministry of Science and Higher Education is gratefully acknowledged.

*rudnicki@cft.edu.pl

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