

## Dynamics of Skyrmion Crystals in Metallic Thin Films

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We study the collective dynamics of the Skyrmion crystal in thin films of ferromagnetic metals resulting from the nontrivial Skyrmion topology. It is shown that the current-driven motion of the crystal reduces the topological Hall effect and the Skyrmion trajectories bend away from the direction of the electric current (the Skyrmion Hall effect). We find a new dissipation mechanism in noncollinear spin textures that can lead to a much faster spin relaxation than Gilbert damping, calculate the dispersion of phonons in the Skyrmion crystal, and discuss the effects of impurity pinning of Skyrmions.

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*Introduction.*—A Skyrmion is a topologically nontrivial soliton solution of the nonlinear sigma model. It was noted early on that Skyrmions in three spatial dimensions have physical properties of baryons and the periodic Skyrmion crystal (SkX) configurations were used to model nuclear matter [1,2]. Skyrmions in two spatial dimensions play an important role in condensed matter systems, such as quantum Hall ferromagnets [3,4]. It was suggested that SkX configurations can be stabilized by the Dzyaloshinskii-Moriya (DM) interaction in ferromagnets without inversion symmetry [5]. Such a state was recently observed in a neutron scattering experiment in the *A* phase of the ferromagnetic metal MnSi [6].

Recent Monte Carlo simulations indicated much greater stability of the SkX when a bulk ferromagnet is replaced by a thin film [7]. This result was corroborated by the real-space observation of SkX in Fe<sub>0.5</sub>Co<sub>0.5</sub>Si thin film in a wide magnetic field and temperature range [8]. Lorentz force microscopy showed that Skyrmions form a triangular lattice with the magnetization vector antiparallel to the applied magnetic field in the Skyrmion center and parallel at the periphery, as was also concluded from the neutron experiment [6].

The next important step is to explore dynamics of Skyrmion crystals and the ways to control them in analogy to the actively studied current- and field-driven motion of ferromagnetic domain walls [9]. Recent observation of the rotational motion of the SkX in MnSi suggests that Skyrmions can be manipulated by much smaller currents than domain walls [10].

In this Letter we study the coupled dynamics of spins and charges in the SkX, focusing on effects of the nontrivial Skyrmion topology and effective gauge fields induced by the adiabatic motion of electrons in the SkX. We derive equations of motion for the collective variables

describing the SkX, calculate its phonon dispersion, and discuss a new form of damping, which can be the dominant spin-relaxation mechanism in half-metals. In addition, we consider new transport phenomena, such as the topological Hall effect in a sliding SkX and the Skyrmion Hall effect. We also discuss the Skyrmion pinning by charged impurities and estimate the critical current above which the SkX begins to slide.

*Low-energy excitations in Skyrmion crystals.*—An isolated Skyrmion has two zero modes corresponding to translations along the *x* and *y* directions. Since an applied magnetic field opens a gap in the continuum of spin-wave excitations, the low-energy magnetic modes in SkX are expected to be superpositions of the Skyrmion displacements, or the phonons. The phonon modes, as well as the coupling of Skyrmion displacements to the external current, can be consistently described in the framework of elasticity theory.

We begin with the spin Hamiltonian  $H_S = \int d^3x [\frac{J}{2a} (\nabla \mathbf{n})^2 + \frac{D}{a^2} \mathbf{n} \cdot [\nabla \times \mathbf{n}] - \frac{h}{a^3} \mathbf{H} \cdot \mathbf{n}]$ , where  $J$  is the exchange constant and  $D$  is the DM coupling that stabilizes the SkX configuration  $\mathbf{n}(\mathbf{x})$  in some interval of the magnetic field  $\mathbf{H} = H\hat{z}$  [5,11]. We calculate the “harmonic lattice energy” by considering a deformation of the SkX,  $\tilde{\mathbf{n}}(\mathbf{x}, t) = \mathbf{n}(\mathbf{x} - \mathbf{u}(\mathbf{x}, t))$ , where the collective coordinate  $\mathbf{u}(\mathbf{x}, t)$  varies slowly at the scale of the SkX lattice constant. The result is

$$H_S = d\eta J \int \frac{d^2x}{\xi^2} [(\nabla u_x)^2 + (\nabla u_y)^2], \quad (1)$$

where  $d$  is the film thickness and  $\xi \sim a\frac{J}{D}$  is the characteristic length scale of the SkX [11], with  $a$  being the lattice spacing. The dimensionless quantity  $\eta = \frac{1}{8\pi} \int_{uc} d^2x (\partial_i \mathbf{n} \cdot \partial_i \mathbf{n})$  encodes the information about  $D$  and  $H$ , and is called the shape factor in what follows.

When an electron current is flowing through the metallic film, the conduction electrons interact with local magnetic moments through the Hund's rule coupling  $H_H = -J_H S \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{n} \psi$ , where  $\psi$  is the electron operator. In the case of small current density and the Skymion size much larger than the Fermi wavelength of conduction electrons, one can apply the adiabatic approximation in which the electron spins align perfectly with the local moment.  $\psi$  is projected into the fully polarized state by  $\psi = \chi |\mathbf{n}\rangle$  with  $\boldsymbol{\sigma} \cdot \mathbf{n} |\mathbf{n}\rangle = |\mathbf{n}\rangle$ . Then the electron action  $S_{\text{el}} = \int dt d^3x [i\hbar \psi^\dagger \dot{\psi} + \frac{\hbar^2}{2m} \psi^\dagger \nabla^2 \psi + J_H S \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{n} \psi]$  can be rewritten as  $S_{\text{el}} = \int dt d^3x [i\hbar \chi^\dagger \dot{\chi} - e a_0 - \frac{1}{2m} \chi^\dagger (-i\hbar \nabla - \frac{e}{c} \mathbf{a})^2 \chi + J_H S \chi^\dagger \chi]$ , where  $a_\mu = \frac{\hbar c}{2e} (1 - \cos\theta) \partial_\mu \varphi$  with  $\theta$  and  $\varphi$  being the spherical angles describing the direction of the local magnetization [12,13]. The gauge potential  $a_\mu$  gives rise to internal electric and magnetic fields,  $\mathbf{e}$  and  $\mathbf{h}$ , acting on spin-polarized electrons passing through the SkX in analogy with the electromagnetic gauge field. Crucially, the internal magnetic field  $\mathbf{b} = \nabla \times \mathbf{a} = \frac{\hbar c}{2e} (\mathbf{n} \cdot \partial_x \mathbf{n} \times \partial_y \mathbf{n}) \hat{z}$  is intimately related to the topological charge  $Q$  of Skymions by [14]  $Q = \frac{1}{4\pi} \int_{\text{uc}} d^2x (\mathbf{n} \cdot \partial_x \mathbf{n} \times \partial_y \mathbf{n}) = \pm 1$ , where the integration goes over the unit cell of the SkX. In the language of internal gauge field, this topological feature is nothing but the quantization of internal flux  $\Omega = \int \mathbf{h} \cdot d\mathbf{S}$  in units of  $\hbar c/e$ . The coupling of the electric current to the internal gauge field induced by the SkX,  $H_{\text{int}} = -\frac{1}{c} \int d^3x \mathbf{j} \cdot \mathbf{a}$ , has a simple form in terms of the collective coordinates introduced above:

$$H_{\text{int}} = d \frac{\hbar Q}{e} \int \frac{d^2x}{\xi^2} (u_x j_y - u_y j_x). \quad (2)$$

The crucial difference between the SkX and a conventional crystal is the form of kinetic energy. The spin dynamics originates from the Berry phase (BP) action,  $S_{\text{BP}} = \frac{d}{\gamma} \int dt d^2x (\cos\theta - 1) \dot{\varphi}$ . Here,  $\gamma = \frac{a^3}{\hbar(S+x/2)}$ , where  $x$  is the filling of the conduction band, and  $S + x/2$  is the total spin average per lattice site. In terms of  $\mathbf{u}$  the kinetic energy has the form

$$S_{\text{BP}} = \frac{dQ}{\gamma} \int dt \frac{d^2x}{\xi^2} (u_x \dot{u}_y - u_y \dot{u}_x). \quad (3)$$

This form of the Berry phase shows that the collective variables  $u_x$  and  $u_y$  describing local displacements of Skymions form a pair of canonical conjugate variables, replacing  $\cos\theta$  and  $\varphi$ . This characteristic property of SkX leads to several unusual responses to applied electric currents and fields. It originates from the Skymion topology and distinguishes SkX from nontopological spin textures such as spirals and domain wall arrays.

Using Eqs. (1)–(3), we obtain the equation of motion for  $\mathbf{u}$ :

$$\dot{\mathbf{u}} = -\frac{e\hbar\gamma}{2} \mathbf{j} + Q \frac{\gamma\eta J}{e\hbar} \hat{z} \times \nabla^2 \mathbf{u}. \quad (4)$$

Two consequences follow immediately. First, the dispersion of phonons in the SkX obtained from Eq. (4) is quadratic,

$$\hbar \omega = \frac{\eta J a^2}{(S + \frac{x}{2})} k^2, \quad (5)$$

in contrast to the linear phonon dispersion in usual crystals and similar to the dispersion of magnons in a uniform ferromagnet. Since  $u_x$  and  $u_y$  play the role of the coordinate and momentum, the longitudinal and transverse phonon modes in the SkX merge into a single mode corresponding to the rotational motion of Skymions, which leads to the quadratic dispersion. Secondly, the SkX can move as a whole driven by the charge current  $\mathbf{j}$ , with a velocity  $\mathbf{V}_{\parallel} = \dot{\mathbf{u}} = -\frac{e\hbar\gamma}{2} \mathbf{j}$ . This rigid motion of SkX leads to several interesting results discussed below.

*Hall effect due to SkX motion.*—In such nontrivial spin textures, the external magnetic field (less than 0.2 T for MnSi) is more than 1 order of magnitude smaller than the internal one, so that it would be neglected in what follows. As can be seen from Eq. (2), the collective coordinates  $u_x$  and  $u_y$  play the role of electromagnetic gauge potentials  $A_y$  and  $-A_x$ , respectively. It is thus expected that the temporal variation of  $\mathbf{u}$  induced by the current leads to a transverse potential drop. This Hall-type effect can also be intuitively understood using the internal magnetic field  $\mathbf{b}$  introduced above. A moving spin texture  $\mathbf{n}(\mathbf{x} - \mathbf{V}_{\parallel} t)$  induces an internal electric field  $\mathbf{e}$  analogous to the electric field of a moving magnetic flux and related to the internal magnetic field by  $\mathbf{e} = -\frac{1}{c} [\mathbf{V}_{\parallel} \times \mathbf{b}]$ . For SkX with  $\mathbf{b} = b_z \hat{z}$ , this electric field generates an electric current in the direction transverse to  $\mathbf{V}_{\parallel}$  resulting in the Hall conductivity:

$$\frac{\Delta\sigma_{xy}}{\sigma_{xx}} \approx -\frac{x}{2S+x} \frac{e\langle b_z \rangle \tau}{mc}, \quad (6)$$

where  $m$  is the electron mass and  $\tau$  is the relaxation time. The average internal magnetic field is  $\langle b_z \rangle = \frac{Q\Phi_0}{2\pi\xi^2}$ , where  $\Phi_0$  is the elementary flux and  $2\pi\xi^2$  is the area of the unit cell of the SkX. This Hall conductivity has the same order of magnitude as the one resulting from the so-called topological Hall effect observed in a *static* SkX [15]. The latter effect is nothing but the Hall effect induced by  $\mathbf{b}$  via  $\mathbf{e} = \frac{1}{c} [\mathbf{v} \times \mathbf{b}]$ , and  $\sigma_{xy}^{\text{Top}}/\sigma_{xx} \approx e\langle b_z \rangle \tau/mc$ , where  $\mathbf{v}$  is the electron velocity. Our new effect differs by the factor of  $-\frac{x}{2S+x}$  from the topological Hall effect. Its physical origin can be easily understood by noting that the total force acting on a single conduction electron is  $\mathbf{F} = -\frac{e}{c} \times [(\mathbf{v} - \mathbf{V}_{\parallel}) \times \mathbf{b}]$ , i.e., the Lorentz force on electrons due to the internal magnetic field of the SkX depends on the relative velocity of electrons and Skymions. When the SkX begins to slide above the threshold electric current  $j_c$  [16], the net topological Hall voltage will be suddenly reduced by the factor  $\frac{2S}{2S+x}$ , which is how the effect of the

spin-motive force and the collective shift of Skyrmions can be identified experimentally.

*New damping mechanism and Skyrmion Hall effect.*—Previously we have systematically discussed the novel effects related to the internal magnetic field. A natural question thus arises as to whether there is any new phenomena associated with the intrinsic internal electric field, which is  $e_i = -\partial_i a_0 - \frac{1}{c} \dot{a}_i = \frac{\hbar}{2e} (\mathbf{n} \cdot \partial_i \mathbf{n} \times \dot{\mathbf{n}})$ . Because of the time derivative in this expression, its effect is absent in the static spin texture. However, in the present case, the motion of SkX makes it nonvanishing, and leads to an additional current  $\mathbf{j}'$  by  $\mathbf{j}' = \sigma \mathbf{e}$  with  $\sigma$  the conductivity of electrons. Substituting this current into the Landau-Lifshitz-Gilbert equation [12,13]

$$\dot{\mathbf{n}} = \frac{\hbar\gamma}{2e} [\mathbf{j} \cdot \nabla] \mathbf{n} - \gamma \left[ \mathbf{n} \times \frac{\delta H_S}{\delta \mathbf{n}} \right] + \alpha [\dot{\mathbf{n}} \times \mathbf{n}], \quad (7)$$

the time derivative  $\dot{\mathbf{n}}$  receives a correction given by

$$\delta \dot{\mathbf{n}} = \frac{\hbar\gamma\sigma}{2e} (\mathbf{e} \cdot \nabla) \mathbf{n} = \alpha' (\mathbf{n} \cdot \partial_i \mathbf{n} \times \dot{\mathbf{n}}) \partial_i \mathbf{n}. \quad (8)$$

The corresponding dimensionless damping constant is  $\alpha' = \frac{1}{(2S+x)} \frac{a^3 \sigma}{\alpha_{\text{fs}} \xi^2 c}$ , where  $\alpha_{\text{fs}} \approx 1/137$  is the fine structure constant. The time derivative in the right-hand side of Eq. (8) shows that the current induced by the internal electric field leads to dissipation. In contrast to Gilbert, damping this new mechanism does not require relativistic effects and only involves the Hund's rule coupling that conserves the total spin. The relaxation of the uniform magnetization, described by Gilbert damping, is clearly impossible without the spin-orbit coupling, which breaks the conservation of the total spin [17]. This argument, however, does not apply to inhomogeneous magnetic textures where the breaking of the rotational symmetry by noncollinear spin orders enables the relaxation without the spin-orbit coupling (note that  $\alpha'$  vanishes as  $\xi \rightarrow \infty$ ). Despite the nonrelativistic origin,  $\alpha'$  depends on the DM coupling, as the latter determines the Skyrmion size. Estimates of  $\alpha'$  made below show that in half-metals it can greatly exceed  $\alpha$ .

The effect of this new dissipation can be observed by tracing the trajectory of Skyrmion motion. Including the new dissipation term, the modified equation of motion (4) for the rigid collective coordinates  $\mathbf{u}(t)$  has the form

$$\dot{\mathbf{u}} = -\frac{e\hbar\gamma}{2} \mathbf{j} - Q(\alpha\eta + \alpha'\eta') \hat{\mathbf{z}} \times \dot{\mathbf{u}}, \quad (9)$$

where the second shape factor  $\eta'$  is given by  $\eta' = \frac{Q}{4\pi} \int_{\text{uc}} d^2x (\mathbf{n} \cdot \partial_x \mathbf{n} \times \partial_y \mathbf{n}) (\partial_i \mathbf{n} \cdot \partial_i \mathbf{n}) / \int_{\text{uc}} d^2x (\partial_i \mathbf{n} \cdot \partial_i \mathbf{n})$ . The new dissipation term in Eq. (9) is obtained by multiplying Eq. (8) with  $\partial_j \mathbf{n}$ , using  $\dot{\mathbf{n}} = -(\dot{\mathbf{u}} \cdot \nabla) \mathbf{n}$ , and integrating over one unit cell. The whole damping term leads to a transverse motion with velocity

$$\mathbf{V}_{\perp} \approx Q(\alpha\eta + \alpha'\eta') [\mathbf{V}_{\parallel} \times \hat{\mathbf{z}}]. \quad (10)$$

This Skyrmion Hall effect can be observed by real-space images of Lorentz force microscopy. The corresponding Hall angle is  $\theta = \arctan(\alpha\eta + \alpha'\eta')$ . The estimate given below shows that main contribution to  $\theta$  comes from the new dissipation mechanism.

*Pinning of Skyrmion crystal.*—Next we consider the pinning of the SkX by charged impurities. The pinning results from spatial fluctuations of the impurity density and variations of the spin direction in the SkX. Variations of the density of charged impurities  $\delta n_i$  give rise to local variations of the electron density  $n_e$  and since the double exchange constant  $J$  is proportional to the latter, we have  $\delta J \sim J \delta n_i / n_e$ . The energy per Skyrmion  $E_S \sim Jd/a$ . Denote the number of impurities in this volume by  $N_1$  with  $\langle N_1 \rangle = n_i 2\pi \xi^2 d$  and the variance  $\delta N_1 = \sqrt{N_1}$ , we obtain the typical variation of the Skyrmion energy:

$$V_1 = \delta J \frac{d}{a} \sim \frac{J}{n_e 2\pi \xi^2 a} \sqrt{N_1} = \frac{J}{n_e a \xi} \sqrt{\frac{n_i d}{2\pi}}. \quad (11)$$

The potential energy density is then  $V_0 = V_1 / (2\pi \xi^2)$ . Substituting  $n_i \sim (la^2)^{-1}$ , where  $l$  is the electron mean free path, and  $n_e = \frac{x}{a^3}$ , we obtain  $V_0 \sim \frac{J}{(2\pi)^{3/2} x} \sqrt{\frac{d}{l} \frac{a}{\xi^3}}$ . The pinning regime of the whole SkX depends on the ratio of the pinning energy  $V_1$  and the elastic energy  $E_S$  of a single Skyrmion [18]. Let  $L^2$  be the number of Skyrmions in the domain where  $u \sim \xi$ . The energy gain due to the impurity pinning in the domain is  $\sim -V_1 L$ , while the elastic energy cost  $\sim \frac{Jd}{a}$  is independent of the domain size. Minimizing the total energy per Skyrmion,  $\frac{Jd}{aL^2} - \frac{V_1}{L}$ , we obtain  $L \sim \frac{Jd}{aV_1}$ .  $L \gg 1$  corresponds to the case of weak (or collective) pinning of SkX, while  $L \sim 1$  corresponds to the strong pinning regime.

The pinning potential gives rise to the spin transfer torque  $-Q \frac{\gamma \xi^2}{2d} [\hat{\mathbf{z}} \times \frac{\delta V}{\delta \mathbf{u}}]$  in the right-hand side of Eq. (4). In the steady state of moving SkX this torque has to be compensated by the interaction with the electric current. The critical current density is then

$$j_c \sim \frac{e}{\hbar} \frac{\xi^2}{d} \left\langle \frac{\partial V}{\partial \mathbf{u}} \right\rangle_{\text{steady state}} \sim \frac{e}{\hbar} \frac{\xi V_0}{dL}, \quad (12)$$

in the weak pinning regime, while in the strong pinning case  $L$  has to be substituted by 1. Similarly, one can estimate the gap in the spin-wave spectrum due to the pinning:

$$\hbar \omega_{\text{pin}} \sim \frac{\hbar\gamma\xi^2}{d} \left\langle \frac{\partial^2 V}{\partial \mathbf{u}^2} \right\rangle \sim \frac{\hbar\gamma\xi^2}{d} \frac{V_0 L}{L^2 \xi^2} = \frac{a^3}{dS} \frac{V_0}{L}. \quad (13)$$

*Estimates.*—For estimates we consider MnSi where Mn ions form a (distorted) cubic sublattice with  $a = 2.9 \text{ \AA}$ . The length of the reciprocal lattice vectors of the SkX  $\sim 0.035 \text{ \AA}^{-1}$  corresponds to  $\xi \sim 77 \text{ \AA}$  and  $D \sim 0.1J$ . The kinetic energy scales as  $\hbar^2/m\xi^2 < J_{\text{H}} \sim 1 \text{ eV}$ , so

that the adiabatic approximation is justified. The electron density  $n_e = 3.8 \times 10^{22} \text{ cm}^{-3}$  [15] corresponds to  $x = n_e a^3 \sim 0.9$  charge carriers per lattice site, while the residual resistivity  $\rho \sim 2\mu\Omega \cdot \text{cm}$  [15] gives an estimate of the impurity concentration  $x_i \sim 5 \times 10^{-3}$ . From the magnon dispersion in the spiral state [19],  $J \sim 3 \text{ meV}$ .

Using these parameters we get  $\alpha' \sim 0.1$ , which shows that the damping resulting from the electric currents generated by noncollinear spin textures can be the dominant mechanism of spin-relaxation in half-metals, where the Gilbert damping constant  $\alpha$  is 1–3 orders of magnitude smaller [20,21]. We note, however, that since the typical electron mean free path  $\xi \sim 500 \text{ \AA}$  is larger than the Skyrmion size  $\xi$ , the relation between the current and internal electric field is nonlocal. The topological Hall angle,  $\theta_H \sim \frac{e\hbar_z\tau}{mc} = \frac{1}{\alpha_{st}n_e\xi^2} \frac{\sigma}{c}$  at  $T = 0$  and the change in  $\theta_H$  induced by the sliding Skyrmion crystal [see Eq. (6)] are also  $\sim 0.1$ . For a 10 nm thick film the parameter  $L \sim x \frac{\xi\sqrt{2\pi dl}}{a^2} \sim 10^3$ , i.e., the pinning of Skyrmions by charged impurities is exceedingly weak and the corresponding values of the pinning frequency  $\hbar\omega_{\text{pin}} \sim 5 \times 10^{-11} \text{ meV}$  and the critical current  $j_c \sim 0.2 \text{ A} \cdot \text{cm}^{-2}$ . This value is much lower than that for domain walls [9] and smaller than the ultralow threshold current  $j_c \sim 10^2 \text{ A} \cdot \text{cm}^{-2}$  observed in bulk MnSi [10], which may be attributed to other pinning mechanisms and the different dimensionality of the system. This low current density also justifies the adiabatic approximation applied in this work.

*Comparison with vortex dynamics.*—Finally, we compare the dynamics of SkX with that of the vortex lines (VL) in type II superconductors. A similar quadratic dispersion was obtained for VL [22]. However, in superconductors it results from long-range interactions between vortices, while the interactions between Skyrmions are short-range (if one ignores the relatively weak dipole-dipole interactions). The absence of long-range interactions in SkX ensures the stability of the quadratic dispersion. Furthermore, the kinetic terms in VL and SkX are completely different. For VL  $u_x$  and  $u_y$  are two independent variables, while for SkX they are conjugated variables as in Eq. (3). Therefore VL are massive, while Skyrmions are not. When a supercurrent flows through the type II superconductor, the charged Cooper pairs are deflected by the VL through the Lorentz force, which in turn gives rise to the transverse motion of the VL. The VL dynamics is usually assumed to be overdamped [23], the kinetic energy of VL is neglected, and the Lorentz force is assumed to be counterbalanced by the friction force. In contrast, the damping of Skyrmions is relatively weak. The spin torque resulting from the strong Hund's rule coupling results in a nearly longitudinal Skyrmion motion. The Hall motion of VL results in a longitudinal voltage drop, which is not important for SkX motion due to the small Hall angle.

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*Note added.*—During the completion of this Letter we became aware of a recent paper by Kim and Onoda addressing the dynamics of Skyrmions in an itinerant double-exchange ferromagnet using a Chern-Simons-Maxwell approach [24]. Their focus, however, seems to differ from ours.

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