Envelope Quasisolitons in Dissipative Systems with Cross-Diffusion

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We consider two-component nonlinear dissipative spatially extended systems of reaction-cross-diffusion type. Previously, such systems were shown to support "quasisoliton" pulses, which have a fixed stable structure but can reflect from boundaries and penetrate each other. Herein we demonstrate a different type of quasisolitons, with a phenomenology resembling that of the envelope solitons in the nonlinear Schrödinger equation: spatiotemporal oscillations with a smooth envelope, with the velocity of the oscillations different from the velocity of the envelope.

Introduction.—Dissipative structures, i.e., patterns in spatially extended systems away from equilibrium, have been intensively studied for many decades. A very comprehensive review can be found in Cross and Hohenberg [1]; results obtained since then would probably require an even more extensive review. A very popular class of mathematical models is the reaction-diffusion systems with diagonal diffusion matrices. There have been numerous indications that nondiagonal elements in diffusion matrices, i.e., cross-diffusion, can lead to new nontrivial effects not observed in classical reaction-diffusion systems, e.g., *quasisolitons*, in systems with an excitable reaction part [2,3]. The defining features of the quasisolitons was their ability to penetrate each other, which makes them akin to the true solitons in the conservative systems. However, the question remained whether this similarity is a reflection of common mechanisms or is entirely superficial and incidental. Here we report an observation which makes the similarity even more striking. Namely, the previously reported quasisolitons propagated while retaining a fixed shape profile, i.e., were constant solutions in a comoving frame of reference; the exceptions were the "aging" quasisolitons reported in [3] which retained their front and tail structures but changed their overall length. Here we report "envelope quasisolitons" (EQS), which share some phenomenology with envelope solitons in the nonlinear Schrödinger equation (NLS) [1,4]. Namely, they have the form of spatiotemporal oscillations ("wavelets") with a smooth envelope, and the velocity of the individual wavelets (the phase velocity) is different from the velocity of the envelope (the group velocity). This may be serious evidence for some deep relationship between these phenomena from dissipative and conservative realms.

Our observations are made in two two-component models, supplemented with cross-diffusion, rather than self-diffusion terms; such terms may appear, say, in mechanical [5], chemical [6], or ecological ([7], p. 11) applications:

$$\frac{\partial u}{\partial t} = f(u, v) + \frac{\partial^2 v}{\partial x^2}, \qquad \frac{\partial v}{\partial t} = g(u, v) - \frac{\partial^2 u}{\partial x^2}. \quad (1)$$

We consider the FitzHugh-Nagumo (FHN) kinetics, taken in the form

$$f = u(u - a)(1 - u) - k_1 v, \qquad g = \varepsilon u, \quad (2)$$

as an archetypal excitable model, with an arbitrarily fixed value of parameter $k_1 = 10$, and varied values of parameters a and ε. As a specific example of a real-life system, we also consider the Lengyel-Epstein (LE) [8] model of a chlorite-iodide-malonic acid-starch autocatalitic reaction system,

$$f = A - u - \frac{4uv}{1 + u^2}, \qquad g = B\left(u - \frac{uv}{1 + u^2}\right), \quad (3)$$

for fixed $A = 6.3$ and $B = 0.055$. We simulated (1) on an interval $x \in [0, L]$, $L \leq \infty$, with Neumann boundary conditions for both u and v [9].

Figure 1 illustrates development and subsequent propagation of an EQS solution in (1) and (2). The profiles are presented in a comoving frame of reference, with the x coordinate measured with respect to the center of mass x_c of the quasisoliton [9]. Simulations with different initial conditions show that as long as the initial perturbation is above a threshold, the amplitude and overall shape of the quasisoliton does not depend on its details. An important feature, evident from comparing 1(d) and 1(e), is that, whereas the overall shape (the envelope) of the quasisoliton and the wavelength of the high-frequency oscillations (the wavelets) within that envelope remain unchanged, the phase of the wavelets relative to the envelope position changes, so the phase velocity (of the wavelets) is different from the group velocity (of the envelope).

This feature can also be seen in Fig. 2(b). The thin stripes in the density plot represent individual wavelets, and the broader band, consisting of these stripes, represents
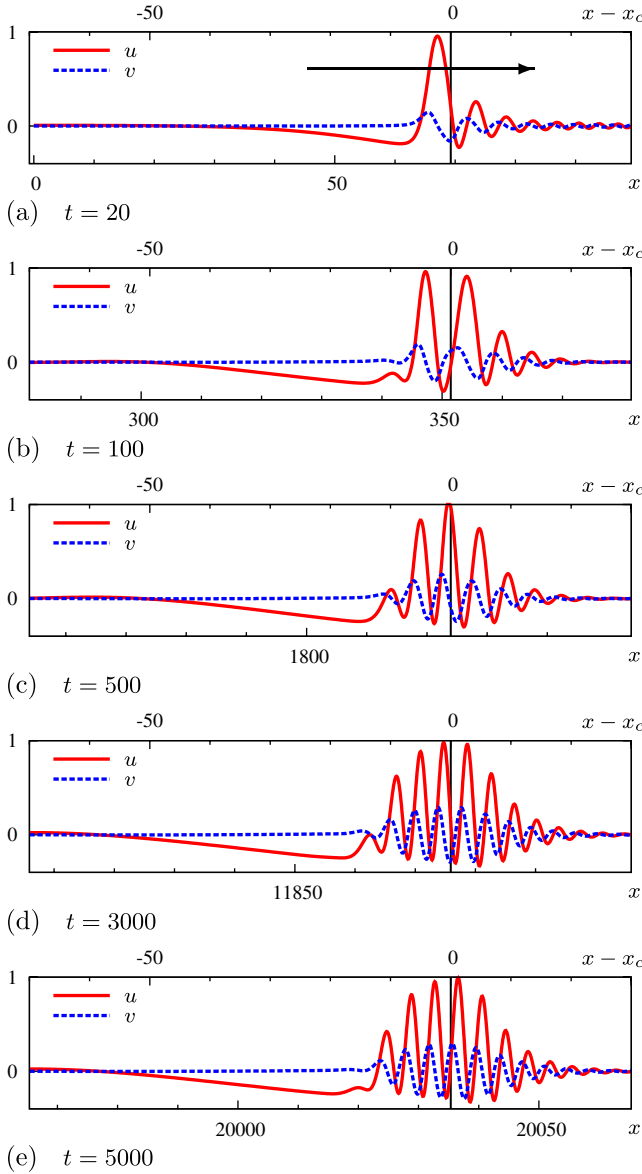
FIG. 1 (color online).   Quasisoliton profiles at the indicated moments of time, in a comoving frame of reference, upper x axes, with the reconstructed original spatial coordinates shown on lower x axes. Parameters $a = 0.12$, $\varepsilon = 0.01$, $L = \infty$, solution propagates rightwards. (a)–(c) Development of a quasisoliton. (d),(e) Propagation of a quasisoliton, with unchanged envelope and shifting phase of high-frequency wavelets within the envelope. Note that wavelets in (d) and (e) are in antiphase: the v profile at $x = x_c$ is near a local minimum in (d) and a local maximum in (e).

the envelope. The slope of the stripes is the inverse of the phase velocity, and the slope of the band is the inverse of the group velocity. The stripes are not parallel to the band because the group and the phase velocities differ. This figure also illustrates another important phenomenon: the reflection of the quasisoliton from the boundary.

Figures 2(a) and 2(c) illustrate two different sorts of solutions which are observed at higher and lower values of parameter a, which also reflect from the boundary.

In Fig. 2(a), we still see individual wavelet stripes that are not parallel to the envelope bands, but there are two bands in the incident wave. Note that the reflected wave only has one band; however, if that reflected band is allowed to propagate for a sufficiently long time, it will spawn its twin band behind it. This is a "multiplying" EQS. We do not go further into properties of these solutions, reserving that for a future study.

In Fig. 2(c) there is only one dominant stripe and many weaker stripes, all of which are parallel to the band. This solution has phenomenological features similar to quasisolitons described previously, e.g., in [2], namely, the wave retains constant shape as it propagates, and reflects from a boundary.

Figure 2(d) shows a quasisoliton reflecting from the boundary, in the other model (1) and (3).

Figure 3 gives an overview of the parametric area of the EQS solutions in (1) and (2) and its neighbors. In Fig. 3(b), the parameter sets at which EQS solutions like the one shown in Fig. 2(b) have been observed are designated by red solid circles. This area is surrounded (i) at higher and lower values of ε, by solutions which have similar shape to those shown in Figs. 1 and 2(b), but do not reflect from boundaries ("annihilating," blue crosses), (ii) at lower values of a, by multiplying EQSs, one of which is illustrated in Fig. 2(a) ("multiplying," green stars), and (iii) at higher values of a, by constant shape quasisolitons, such as the one shown in Fig. 2(c) ("constant," magenta triangles).

The area of existence of all these solutions in the (a, ε) parametric plane is bounded from above and from the right, and beyond it our initial conditions did not produce any stably propagating solutions ("decaying," black open circles). Figure 3(a) illustrates the variability of the shape of EQSs within their parametric domain. The most important conclusion from Fig. 3 is that the EQSs are not a unique feature of a special set of parameters but are observed in a rather broad parametric area.

The oscillatory character of the fronts of the cross-diffusion waves, described in numerical simulations [2,3] and analyzed theoretically in [2,3,10], was for waves of stationary shape. Although the proper theory of the EQSs is beyond the scope of this Letter, the analysis of their nonstationary front structure is easily achieved via linearization of (1). The resting states in both FHN (2) and LE (3) kinetics are stable foci which already show propensity to oscillations. Considering in more detail the FHN kinetics, for small u, v, the solution has the form

$$\begin{bmatrix} u \\ v \end{bmatrix} \approx \mathrm{Re}(C\mathbf{v}e^{-\mu(x-ct)}e^{i(kx-\omega t)}), \qquad (4)$$

where
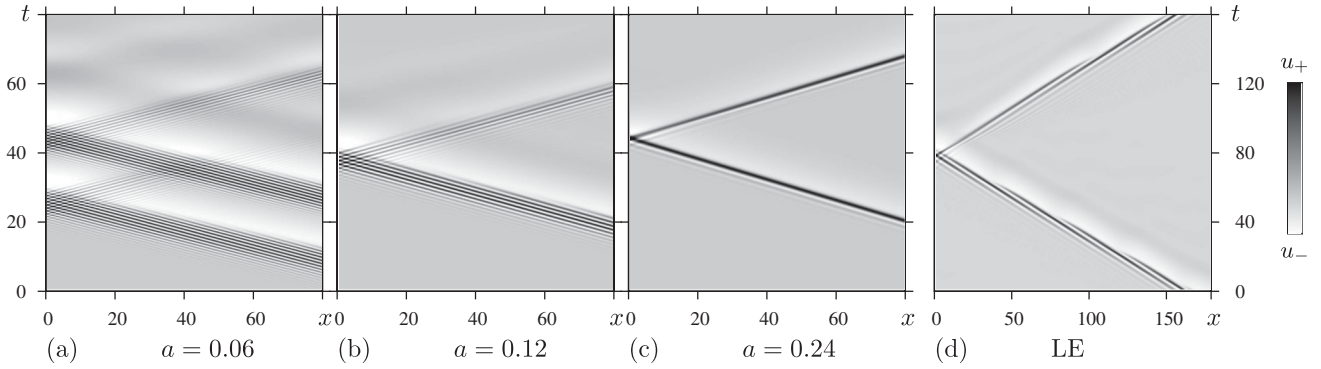
FIG. 2.   Density plots of the quasisolitons reflecting from a boundary [9]. (a)–(c) FHN kinetics, $u_- = -0.3$, $u_+ = 1$, $\varepsilon = 0.01$, and a is varied as shown under the panels. (a) A double EQS becomes a single EQS upon the reflection. Some time after that, it will grow its twin and become a double quasisoliton again. (b) A single EQS: This is the same case as shown in Fig. 1. (c) A "classical" quasisoliton which retains its shape as it propagates. (d) An EQS in the LE kinetics, $L = 300$, $u_- = 0.8$, $u_+ = 3.3$. In all panels, the origin of the t axis is shifted to an arbitrarily chosen moment shortly before the impact event.

$$\mathbf{A}(\lambda, \nu)\mathbf{v} = 0, \qquad \mathbf{v} \neq 0, \qquad \det\mathbf{A} = 0,$$

$$\mathbf{A} = \begin{bmatrix} -a - \lambda & -k_1 + \nu^2 \\ \varepsilon - \nu^2 & -\lambda \end{bmatrix}, \qquad \lambda = \mu c - i\omega, \quad (5)$$

$$\nu = -\mu + ik.$$

Fitting of the v component of a solution shown in Fig. 1 to (4) gives $c \approx 4.07698$, $k \approx 1.71532$, $\mu \approx 0.182305$, and $\omega \approx 6.15190$, which satisfies (5) to 3 significant

FIG. 3 (color online).   (a) The number of wavelets in an EQS as a function of a at fixed $\varepsilon = 0.01$. (b) Areas of different sorts of solutions in the parametric plane (a, ε). The black dashed line corresponds to the parametric cross section shown in (a).

figures [9] The quality of the fitting is illustrated in Fig. 4. Note that the phase velocity of the wavelets here is $c_{ph} = \omega/k \approx 3.59$, smaller than the group velocity, $c \approx 4.08$, which agrees with the fact that the slope of the individual stripes in Fig. 2(b) (which is the inverse of the phase velocity c_{ph}) is steeper than the slope of the band (which is the inverse of the group velocity c).

The shape of the profiles in Fig. 1 is reminiscent of localized states in the generalized Swift-Hohenberg equation with "snakes and ladders" bifurcation diagrams [11]. The essential difference of our solutions is that they move and do not preserve their shape, so they cannot be immediately studied by ordinary differential equations' bifurcation techniques.

The defining features of the EQSs described above are similar to envelope solitons of the nonlinear Schrödinger equation. The version of this equation known as "NLS(+)" [4] can be written in the form

$$i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + w|w|^2 = 0$$

for a complex field w, which presents a reaction-cross-diffusion system for two real fields u and v via $w = u - iv$ of the form (1), with

$$f = u(u^2 + v^2), \qquad g = -v(u^2 + v^2). \quad (6)$$

System (1) and (6) has soliton solutions in the form of (fast) harmonic waves with a unimodal (sech-shaped) envelope, and the propagation velocity of the envelope (the group velocity) is different from the propagation velocity of the wavelets (the phase velocity). Hence one might think of possible interpretation of the EQSs in (1) and (2) or (1) and (3) as a result of a nonconservative perturbation of the envelope solitons in (6), which would select particular values of the otherwise arbitrary amplitude and speed of
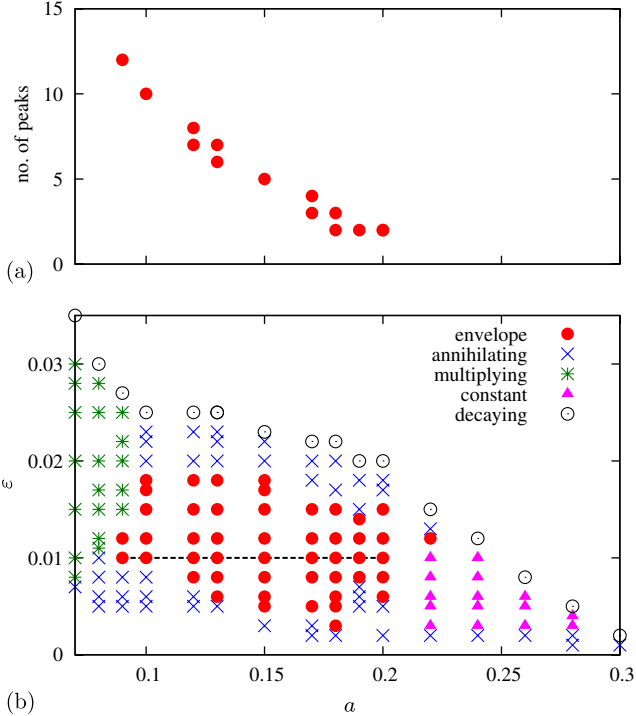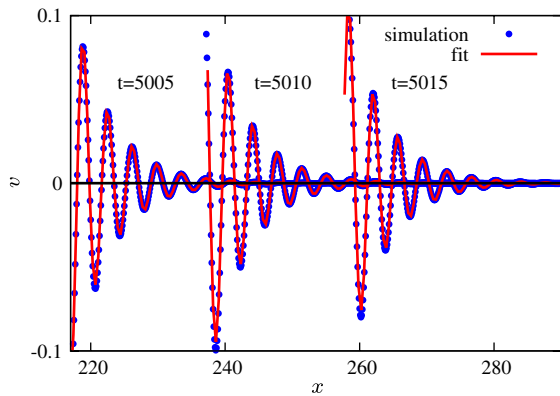
FIG. 4 (color online). Profiles of an EQS wave front and its fitting by (4) at selected moments of time. Parameters are $\varepsilon = 0.01$, $a = 0.12$, $L = \infty$. The origin of the x axis is chosen arbitrarily.

the soliton and modify its shape. This interpretation, however, does not seem to work, and our attempts to connect the solutions in (1) and (6) and (1) and (2) via a one-parametric family of systems have been unsuccessful, as the EQS solutions disappeared during parameter continuation. The apparent reason is that the sense of rotation of solutions of (6) in the (u, v) is clockwise, whereas in (2) and (3) it is counterclockwise, and the variant of (6) with counterclockwise rotation, the "NLS(−)" equation, does not have envelope soliton solutions.

Another comparison can be made with "wave packets" reported by Vanag and Epstein in microemulsion Belousov-Zhabotinsky reaction, and corresponding mathematical models, associated with finite-wavelength instability of an equilibrium in a reaction-diffusion system with unequal self-diffusion coefficients [12]. They considered two distinct types of solutions: small- and large-amplitude wave packets (SAWP and LAWP), both capable of reflection from boundaries. SAWP are observed in the nearly linear regime; they have phase speed (of the wavelets) different from group speed (of the envelope, or the packet). However, being near to a linear instability and having no stabilizing effect of the dispersion as in NLS(+), the packets slowly grow both in amplitude and in width; i.e., they are not quasisolitons. The LAWP, on the contrary, have fixed amplitude and width, but their phase and group velocities coincide, so they retain constant shape. They are therefore phenomenologically similar to the quasisolitons reported in excitable systems with cross-diffusion [2]. Note that adiabatic elimination of a fast component in a reaction-diffusion system with very different self-diffusion coefficients is one of the ways in which cross-diffusion

terms may appear [6,13], so this analogy deserves further investigation.

To conclude, the solutions we have reported resemble the NLS(+) envelope solitons by their morphology and by their ability to reflect from boundaries; however, they are different in that the amplitudes and speeds of NLS solitons depend on initial conditions, while in (1) they are fixed by parameters of the models. The reported solutions are similar to quasisolitons reported earlier in that they share the fixed amplitude and reflection properties, but different in that they do not preserve constant shape as their phase velocities are different from their group velocities. Hence we believe this is a new nonlinear phenomenon not seen before. The mechanisms behind the key properties of this new type of solutions require further investigation; however, it is already clear that this is not simply a nonconservative perturbation of the NLS.

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