Amplifying Single-Photon Nonlinearity Using Weak Measurements

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We show that weak measurement can be used to ''amplify'' optical nonlinearities at the single-photon level, such that the effect of one properly postselected photon on a classical beam may be as large as that of many unpostselected photons. We find that ''weak-value amplification'' offers a marked improvement in the signal-to-noise ratio in the presence of technical noise with long correlation times. Unlike previous weak-measurement experiments, our proposed scheme has no classical equivalent.

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An interaction between two independent photons could be used to serve as a ''quantum logic gate,'' enabling the development of optical quantum computers [1–3], as well as opening up an essentially new field of quantum nonlinear optics [4]. Typical optical nonlinearities are many orders of magnitude too weak to create a π phase shift as required in initial proposals, but more recently it was realized that any phase shift large enough to be measured on a single shot could be leveraged into a quantum logic gate [5]. Much recent work has shown that atomic coherence effects [6–9] and nonlinearities in microstructured fiber [10,11] can generate greatly enhanced Kerr nonlinearities. While even a very small phase shift can be made larger than the quantum (shot) noise, by using a sufficiently intense probe, present experiments are limited by technical rather than quantum noise and difficult to carry out even with much averaging. For example, in Ref. [11], a phase shift of 10^{-7} rad was measured by averaging over 3×10^9 classical pulses with single-photon-level intensities. To date, no one has yet been able to observe the cross-Kerr effect induced by a single propagating photon on a second optical beam [12]. In this Letter, we show that using weak-value amplification (WVA) [13–15], a single photon can be made to ''act like'' many photons, and it is possible to amplify a cross-Kerr phase shift to an observable value, much larger than the intrinsic magnitude of the single-photon-level nonlinearity. In so doing, we also demonstrate quantitatively how WVA may improve the signal-to-noise ratio (SNR) in appropriate regimes, a result of broad general applicability to quantum metrology.

Weak measurement is an exciting new approach to understanding quantum systems from a time-symmetric perspective, obtaining information from both their preparation and subsequent postselection [16]. In the past several years, it has been widely studied to address foundational questions in quantum mechanics [17], as well as for its potential application to ultrasensitive measurements [14,15,18,19]. If a quantum system is coupled only weakly to a probe, then very little information may be obtained from a single measurement, and in compensation, this measurement disturbs the system by a negligible amount. In such situations, if the system is prepared in some initial state $|i\rangle$ and postselected in some other final state $|f\rangle$, the ''weak value'' $\langle A\rangle_w = \langle f|A|i\rangle/\langle f|i\rangle$ describes the mean size of the effect an ensemble of such systems would have on a device designed to measure the observable A. It should be noted that weak values are not guaranteed to lie within the eigenvalue spectrum of the observable A. Specifically, if the overlap between the initial and final states is small, the weak value may be anomalously large. In Aharonov, Albert, and Vaidman's famous example, the spin of an electron may be measured to be 100 [13]; in a mathematically equivalent sense, we show that the effective photon number in one arm of an interferometer may be found to be 100 even if the entire interferometer contains only one photon.

Unfortunately, WVA always comes at the cost of reducing the sample size (via postselection) by just enough to nullify any potential improvement in SNR, at least in the case of statistical noise. Several recent experiments [14,15] observed that many real-world measurements are limited by technical noise, which is not reduced by averaging over more samples, and attempted to show that in such cases weak measurement can indeed be of practical advantage. It still remains unclear exactly when such ''technical'' noise could be overcome by using WVA. In Ref. [15(b)], a very specific noise model was assumed, in which rejection of photons through postselection did not reduce the ultimate signal strength, an assumption we do not make [20]. Here we find that the SNR can be increased, roughly to but not beyond the quantum limit, when the noise correlation times are sufficiently long. Previous weak-measurement demonstrations, instead of entangling a system with a distinct ''probe,'' merely used two degrees of freedom of the same physical photon as the system and probe; this resulted in experiments which could be equally well understood in the framework of classical electromagnetism, with no need of the full quantum formalism of weak measurement. (Some implementations have been carried out with probabilistic coupling between the system and the probe [21].)

Our present proposal demonstrates that two distinct optical beams may be coupled deterministically, by using accessible interactions, in such a way that no classical explanation is possible, that the surprising weak-measurement prediction of a single photon acting like a collection of many photons is rigorously correct, and that the SNR can be substantially improved by WVA, when the noise possesses long correlation times (e.g., $1/f$ noise).

The nonlinear interaction of interest here can be viewed as a measurement in which a single-photon "system" is coupled through the cross-Kerr effect to a classical probe field; see Fig. 1. The single photon is sent through a 50-50 beam splitter, thus prepared in the superposition $|i\rangle \equiv (|b\rangle - |a\rangle)/\sqrt{2}$ of modes a and b. The single photon interacts with a probe through a Kerr medium, leading to a cross-phase shift that we model as $\exp(i\phi_0 \hat{n}_b \hat{n}_c)$, where $\phi_0 \ll 1$ is the cross-phase shift per photon and $\hat{n}_b$ ($\hat{n}_c$) is the number operator for mode b (c). After the interaction with the probe, the system is postselected to be in a state nearly orthogonal to the initial one, $|f\rangle = t|b\rangle + r|a\rangle$, by triggering on the detection of a photon at $D1$. This port exhibits imperfect destructive interference when the reflectivity r and transmissivity t, which we choose to be real and positive, are slightly imbalanced. We define a small postselection parameter $\delta \equiv \langle f|i\rangle = (t-r)/\sqrt{2} \ll 1$. The weak value of the photon number in mode b is given by

$$\langle \hat{n}_b \rangle_w = \frac{\langle f|\hat{n}_b|i\rangle}{\langle f|i\rangle} = \frac{t/\sqrt{2}}{(t-r)/\sqrt{2}} \simeq \frac{(1+\delta)/2}{\delta} \simeq \frac{1}{2\delta}. \quad (1)$$

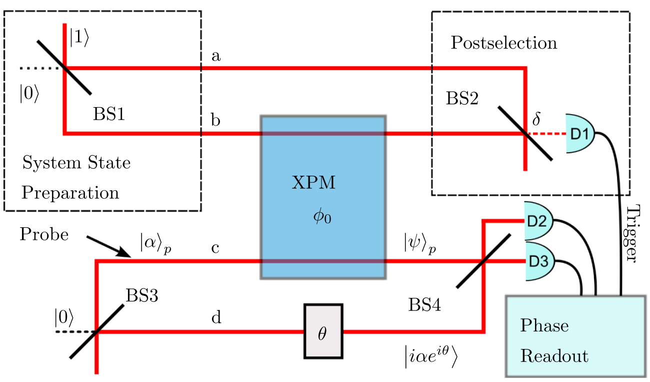

FIG. 1 (color online). The single-photon system is prepared in an equal superposition of arms a and b by the first beam-splitter (BS1). After a weak cross-phase-modulation (XPM) interaction with the probe, prepared in a coherent state $|\alpha\rangle_p$, the system is postselected on a nearly orthogonal state by detecting the single photon in the nearly dark port, $D1$. The success probability of postselection depends on the imbalance δ in the reflection and transmission coefficients of the beam splitter BS2 and the backaction of the probe on the system. Using the lower interferometer to read out the phase shift of the probe amounts to a measurement of the system observable n_b, the photon number in arm b. The phase shifter θ is used to maximize the sensitivity of the measurement.

This means that whenever the postselection succeeds (which occurs with probability δ^2 ignoring the measurement backaction) the weak value of the photon number in mode b is $1/\delta$ times the strong value $1/2$. The postselection parameter δ can be very small, leading to a large weak value for the photon number in the system. Therefore, within the weak-measurement formalism, the probe will experience a cross-phase shift equivalent to that of many photons, even though the system never has more than one photon. In the rest of this Letter, we will show explicitly that such a scheme does in fact lead to a large phase shift and quantify the improvement in the SNR as a function of the characteristics of the technical noise.

The state of the system and probe after coupling is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|b\rangle_s|\alpha e^{i\phi_0}\rangle_p - |a\rangle_s|\alpha\rangle_p). \quad (2)$$

For $\phi_0 \ll 1$, the overlap between the two possible final probe states is $\langle\alpha|\alpha e^{i\phi_0}\rangle \simeq e^{i|\alpha|^2\phi_0 - |\alpha|^2\phi_0^2/2}$. The amplitude of this overlap, $e^{-|\alpha|^2\phi_0^2/2}$, has to be close to 1 for the interaction to be weak, which implies $|\alpha|\phi_0 \ll 1$. The phase of the overlap, $|\alpha|^2\phi_0$, describes the average phase shift imparted to the system by the probe. This phase does not result in dephasing of the system state and therefore, in principle, can be compensated by adding a phase shifter to the upper interferometer. Without compensation, WVA will occur only when $|\alpha|^2\phi_0$ is close to an integer multiple of 2π, where the overlap between the initial and final states of the system is small. We define ϵ to be the difference between $|\alpha|^2\phi_0$ and the closest multiple of 2π.

If the system is postselected to be in state $|f\rangle$, the state of the probe, $|\psi\rangle_p = {}_s\langle f|\Psi\rangle$, collapses to a superposition of two coherent states:

$$|\psi\rangle_p = \sqrt{P^{-1}}\,\tfrac{1}{2}[(1+\delta)|\alpha e^{i\phi_0}\rangle - (1-\delta)|\alpha\rangle], \quad (3)$$

where $P \simeq |\alpha|^2\phi_0^2/4 + \delta^2 + \epsilon^2/4$ is the post-selection probability. The final state of the probe can be most easily understood by displacing it to the origin in phase space, defining $|\chi\rangle = D^\dagger(\alpha)|\psi\rangle_p$, where $D(\alpha)$ is the displacement operator. For ϕ_0, $|\alpha|\phi_0 \ll 1$, one can write

$$|\chi\rangle \simeq \sqrt{P^{-1}}[(\delta + i\epsilon/2)|0\rangle + (i\alpha\phi_0/2)|1\rangle], \quad (4)$$

where $|0\rangle$ and $|1\rangle$ are vacuum and single-photon number states, respectively. The weak-measurement formalism applies if $\delta^2 \gg (\epsilon^2 + |\alpha|^2\phi_0^2)/4$; in particular, as $\epsilon \to 0$, one recovers the weak-measurement prediction $|\psi\rangle_p \simeq |\alpha\exp(i\phi_0/\delta)\rangle$, a coherent state with a largely enhanced phase. On the other hand, if $\delta^2 \ll \epsilon^2/4 + |\alpha|^2\phi_0^2/4$, the postselection is significantly modified by the backaction of the probe on the system. It is instructive to look at both regimes and the transition between them and determine what the maximum possible enhancement is, taking the backaction into account.

Most of the interesting phenomena can be understood by investigating properties of $|\chi\rangle$. If δ or ϵ is much larger than $|\alpha|\phi_0$, then the state $|\chi\rangle$ is approximately equal to a weak coherent state: $|\chi\rangle \simeq |0\rangle + i\alpha\phi_0|1\rangle/(2\delta + i\epsilon)$. It can be seen that δ contributes to a shift in the imaginary quadrature (phase of $|\psi\rangle_p$) and ϵ contributes to a shift in the real quadrature (average photon number). On the other hand, if $|\alpha|\phi_0$ is much larger than the two other terms, the state $|\chi\rangle$ is approximately a single-photon number state.

The average phase shift can be measured by using the lower interferometer in Fig. 1, e.g., as the ratio of the difference of the photon numbers at $D2$ and $D3$ to the sum:

$$\bar{\phi} = \frac{\langle M_- \rangle_p}{\langle M_+ \rangle_p} \simeq \frac{\delta}{2P}\phi_0, \tag{5}$$

where $M_\pm = \hat{n}_3 \pm \hat{n}_2$. We should compare this value to the phase shift ϕ_0 imparted to the probe by a single photon in path b. The phase that one measures after successful postselection is enhanced by a factor of $\delta/2P$. Figure 2 shows this enhancement factor as a function of postselection parameter δ and the average number of probe photons, $|\alpha|^2$. For sufficiently small backaction, the weak-measurement prediction for the amplification, $1/2\delta$, is correct. However, as δ becomes smaller, the amplification grows but so does the backaction, until at $\delta_{\text{opt}} = \sqrt{|\alpha|^2\phi_0^2 + \epsilon^2}/2$ a maximum amplification value is achieved of $1/4\delta_{\text{opt}}$, half of the weak-measurement value. For small ϵ, the maximum phase shift is equal to $1/2|\alpha|$, which is one-half the quantum uncertainty of the probe phase. Thus, the WVA works up to the point where the single-shot quantum-limited SNR would be on the order of 1. Taking a closer look at the form of state $|\chi\rangle$, one can see that the large phase shift is caused by destructive interference due to postselection; the vacuum term largely cancels out, enhancing the importance of the single-photon term. Note that the large overlap of the two possible probe states corresponding to the two states of the system is essential for this to occur.

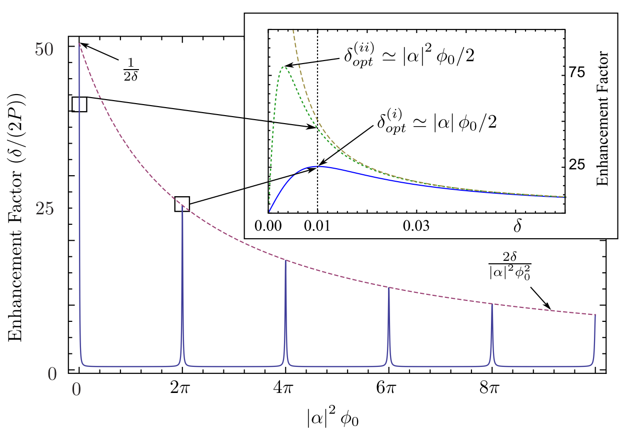

FIG. 2 (color online). The enhancement factor versus $|\alpha|^2\phi_0$. The parameters used are $\phi_0 = 2\pi \times 10^{-5}$ and $\delta = 0.01$. The enhancement factor is calculated by using the state of Eq. (3) without any approximations. The dashed line shows the enhancement factor if the average phase written by the probe on the system, $|\alpha|^2\phi_0$, is compensated; otherwise, enhancement occurs whenever $|\alpha|^2\phi_0$ is close to an integer multiple of 2π (solid curve). The inset shows the enhancement factor as a function of postselection parameter, δ, in two different regimes: (i) $|\alpha|^2 = 10^5$, in which case the imparted phase on the system by the probe, ϵ, is 0 (solid blue line); (ii) $|\alpha|^2 = 10^2$, where ϵ is a small nonzero phase (dashed green line). For large values of δ the weak-measurement prediction is valid; however, as δ decreases, the backaction from the probe plays a more dominant role. The dashed line shows the prediction of the weak-measurement formalism.

The weakness condition $|\alpha|\phi_0 \ll 1$ is often met in experimental situations, either because of the difficulty of approaching quantum-limited performance at high intensities or to avoid additional undesired nonlinear effects. In Ref. [11], for instance, a cross-phase shift of $\phi_0 = 10^{-7}$ rad per photon was reported and unwanted nonlinear effects were observed once the average number of probe photons $|\alpha|^2$ reached about 10^6. In this situation, both conditions of $|\alpha|\phi_0 \ll 1$ and $|\alpha|^2\phi_0 \ll 1$ are met, and WVA can be used to enhance the SNR.

In practice, phase measurement is subject to both quantum and technical noise. While the average measured phase is enhanced by a factor of $\delta/2P$, we expect the uncertainty due to statistical noise to be inversely proportional to the square root of the sample size, thus scaling as $1/\sqrt{P}$ (recall that P is the probability of successful postselection). The overall SNR is hence multiplied by a factor $\delta/2\sqrt{P}$, which has a maximum value of 1/2 (the *actual* photon number in arm b); in the case of pure quantum noise, for instance, there is no advantage with postselection. In what follows, using a more general noise model, we study under what type of technical noise WVA can be beneficial.

Consider a nonpostselected measurement performed over a total time T. Single photons are sent to the upper interferometer at a rate Γ, and phase measurement is triggered by the detection of a single photon. We term the outcome of the ith measurement $\phi_m^i = \bar{\phi} + \eta^i$, where the zero-mean fluctuating term η^i includes the quantum and technical noise. The average measured phase shift is $\phi_m = 1/(\Gamma T)\sum_{i=1}^{\Gamma T}\langle \phi_m^i \rangle = \bar{\phi}$. The uncertainty in this average value is given by $(\Delta\phi_m)^2 = 1/(\Gamma T)^2 \sum_{i,j=1}^{\Gamma T}\langle \eta^i \eta^j \rangle$. There are two possible extremes to be considered. In the white-noise limit (noise correlation time τ_c much shorter than the mean time between successive measurements, $1/\Gamma$), the correlation function can be modeled as a delta function: $\langle \eta^i \eta^j \rangle = \bar{\eta}^2 \delta_{ij}$. In particular, this holds for quantum (shot) noise. In this limit the noise scales statistically with the number of measurements: $\Delta\phi_m = \bar{\eta}/\sqrt{\Gamma T}$. The opposite extreme is that of noise with

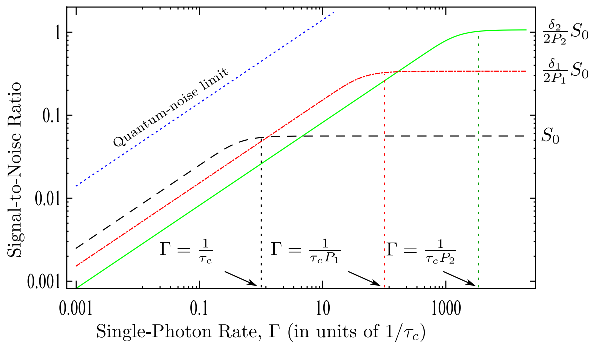

FIG. 3 (color online). The SNR as a function of the single-photon rate Γ. The technical noise is modeled by an exponential correlation function with an amplitude, $\bar{\eta}$, 10 times larger than the quantum noise. The dashed line shows the nonpostselected SNR for the phase shift due to one photon in mode b. The postselected SNR for $\delta_1 = 0.1$ (weak-measurement regime—dash-dotted red line) and $\delta_2 = 0.01$ (the optimum value of measured phase shift—solid green line) are also shown; the dotted line shows the quantum-limited SNR for comparison. The nonpostselected SNR approaches a maximum value S_0 due to low-frequency noise. However, for the postselected SNR, we see enhancement by a factor of $\delta/2P$, compared to the nonpostselected SNR S_0 for measurements with a high enough rate. For low rates the enhancement is given by $\delta/2\sqrt{P}$, and therefore the weak-measurement results in the best possible postselected SNR. We have taken $T/\tau_c = 10^3$, $\phi = 2\pi \times 10^{-5}$, $|\alpha|^2 = 10^5$, and therefore $P_1 = 0.01$ and $P_2 = 3 \times 10^{-4}$.

long-time correlations, $\tau_c \gg 1/\Gamma$, in which case $\langle \eta^i \eta^j \rangle = \bar{\eta}^2$, and averaging cannot help reduce the uncertainty.

In the postselected case, the sample size drops from ΓT to $P\Gamma T$, and $\Delta\phi_m$ increases to $\bar{\eta}/\sqrt{P\Gamma T}$ in the delta-correlated case while it remains constant at $\bar{\eta}$ in the presence of long-time correlations. Given the enhancement factor of $\delta/2P$, the SNR thus scales as $\delta/2\sqrt{P}$ (always <1, as remarked earlier) in the former case but $\delta/2P$ (which may be $\gg 1$) in the latter case.

Figure 3 shows the calculated SNR as a function of single-photon rate, Γ, where the noise is modeled with a correlation function $\langle \eta^i \eta^j \rangle = \delta_{ij}/2|\alpha|^2 + \bar{\eta}^2 \exp(-|i - j|/\Gamma\tau_c)$ to account for delta-correlated quantum noise and a technical contribution with correlation time τ_c. The nonpostselected SNR shows a knee around $\Gamma\tau_c = 1$, separating the regimes where measurements are not correlated ($\Gamma\tau_c \ll 1$) and highly correlated ($\Gamma\tau_c \gg 1$). The SNR has a statistical scaling $\sqrt{\Gamma}$ in the former regime and remains constant in the latter. The graphs for the postselected cases are qualitatively similar, but the knee occurs near $P\Gamma\tau_c = 1$, that is, when the noise in the successive postselected measurements starts to become correlated. Thus whenever the noise exhibits correlations over time scales greater than the mean time between incident photons, the SNR can be improved via postselection.

We have shown that one postselected photon may act like many photons, writing a very large cross-phase shift on a coherent state, and that this amplification may greatly improve the SNR for measuring single-photon-level nonlinearities. Considering presently observable optical nonlinearities, this opens the door to unambiguous weak-measurement experiments, in which two distinct physical systems could be deterministically coupled, leaving no room for an alternative classical explanation. Accounting for the effects of backaction when the weakness criterion is relaxed, we find that the largest achievable phase shift per postselected photon is always of the order of the quantum uncertainty of the probe phase. More generally, we find that although postselection cannot enhance the SNR in the presence of noise with short (or vanishing) correlation times, particularly shot noise, it can be of great use in the presence of noise with long correlation times. Given the prevalence of low-frequency noise (e.g., $1/f$ noise) in real-world systems, this suggests that WVA may find broad application in precision measurement.

During the completion of this work, an independent proposal for weakly coupling photons to atomic ensembles was also posted to the arXiv [22].

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