Superradiant Phase Transitions and the Standard Description of Circuit QED

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We investigate the equilibrium behavior of a superconducting circuit QED system containing a large number of artificial atoms. It is shown that the currently accepted standard description of circuit QED via an effective model fails in an important aspect: it predicts the possibility of a superradiant phase transition, even though a full microscopic treatment reveals that a no-go theorem for such phase transitions known from cavity QED applies to circuit QED systems as well. We generalize the no-go theorem to the case of (artificial) atoms with many energy levels and thus make it more applicable for realistic cavity or circuit QED systems.

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Recent years have seen rapid progress in fabrication and experimental control of superconducting circuit QED systems, in which a steadily increasing number of artificial atoms interact with microwaves [1–4]. These developments set the stage to study collective phenomena in circuit QED. An interesting question in that context is whether a system with many artificial atoms undergoes an equilibrium phase transition as the coupling of artificial atoms and electromagnetic field is increased (at zero temperature). Phase transitions of this type have been intensely discussed for cavity QED systems [5–10] and are known as superradiant phase transitions (SPTs) [6]. However, in cavity QED systems with electric dipole coupling their existence is doubted due to a no-go theorem [8]. Recently, it has been claimed that SPTs are possible in the closely related circuit QED systems with capacitive coupling [10–12]. This would imply that the no-go theorem of cavity QED does not apply and challenges the well-established analogy of circuit and cavity QED.

Here, we show in a full microscopic analysis that circuit QED systems are also subject to the no-go theorem. We argue that such an analysis is necessary since the standard description of circuit QED systems by an effective model (EM) is deficient in the regime considered here. A toy model is used to illustrate this failure of an EM. Finally, we close a possible loophole of the no-go theorem by generalizing it from two-level to multilevel (artificial) atoms. Thus, our work restores the analogy of circuit and cavity QED and rules out SPTs in these systems under realistic conditions that have not been covered before.

Dicke Hamiltonian in cavity and circuit QED.—Both circuit QED systems and cavity QED systems with N (artificial) atoms (Fig. 1) are often described by the Dicke Hamiltonian [13] ($\hbar = 1$)

$$\mathcal{H}_{\mathrm{D}} = \omega a^\dagger a + \frac{\Omega}{2}\sum_{k=1}^{N}\sigma_z^k + \frac{\lambda}{\sqrt{N}}\sum_{k=1}^{N}\sigma_x^k(a^\dagger + a) + \kappa(a^\dagger + a)^2. \quad (1)$$

The (artificial) atoms are treated as two-level systems with energy splitting Ω between ground state $|g\rangle_k = \binom{0}{1}_k$ and excited state $|e\rangle_k = \binom{1}{0}_k$ (σ_x^k, σ_z^k are Pauli matrices). In the case of circuit QED, we assume Cooper-pair boxes as artificial atoms, which justifies the two-level approximation. Our main results, though, hold for any charge-based artificial atoms (capacitive coupling) [14]. Further, $a^\dagger$ generates a photon of energy ω. Matter and field couple with a strength λ. The κ term, often neglected in other contexts, will become crucial below. In cavity QED, $\mathcal{H}_{\mathrm{D}}$ derives from minimal coupling of atoms and electromagnetic field. For an atom (n electrons) at a fixed position,

$$\mathcal{H}^0_{\mathrm{cav}} = \sum_{i=1}^{n}\frac{[\mathbf{p}_i - e\mathbf{A}(\mathbf{r}_i)]^2}{2m} + V_{\mathrm{int}}(\mathbf{r}_1, \ldots, \mathbf{r}_n). \quad (2)$$

The $\mathbf{pA}$ and $\mathbf{A}^2$ terms in the analog N-atom Hamiltonian yield the λ and κ term in $\mathcal{H}_{\mathrm{D}}$, respectively. In circuit QED, $\mathcal{H}_{\mathrm{D}}$ arises from a widely used EM for a charge-based artificial atom in a transmission line resonator [15],

$$\mathcal{H}^0_{\mathrm{cir}} = 4E_C\sum_\nu(\nu - \bar{\nu})^2|\nu\rangle\langle\nu| - \frac{E_J}{2}\sum_\nu(|\nu+1\rangle\langle\nu| + \mathrm{H.c.}).$$

Here, ν counts the excess Cooper pairs on the island, E_J and $E_C = e^2/[2(C_G + C_J)]$ are the Josephson energy and

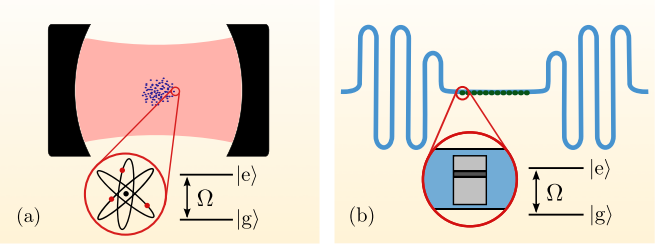

FIG. 1 (color online). Cavity QED system with N atoms (a) and circuit QED system with N Cooper-pair boxes as artificial atoms (b).

the charging energy of the Cooper-pair box, and C_G and C_J are the coupling capacitance and the capacitance of the Josephson junction. Moreover, $\bar{\nu} = C_G(V_G + \mathcal{V})/2e$, V_G is an external gate voltage and $\mathcal{V}$ the quantum voltage due to the electromagnetic field in the resonator. The Cooper-pair box is assumed to be at its degeneracy point [15]. As it is described by macroscopic quantities (like E_C) and only 1 degree of freedom (ν), $\mathcal{H}_0^{\rm cir}$ is an EM for a Cooper-pair box in a transmission line. Starting either from $\mathcal{H}_{\rm cav}^0$ or $\mathcal{H}_{\rm cir}^0$, one obtains $\mathcal{H}_{\rm D}$ using the following approximations: The N (artificial) atoms are identical, noninteracting two-level systems with ground and excited states $|g\rangle$ and $|e\rangle$ which are strongly localized compared to the wavelength of the single considered field mode [i.e., $\mathbf{A}(\mathbf{r}_i^k) \approx \mathbf{A} \equiv A_0\boldsymbol{\epsilon}(a^\dagger + a)$, where $|\boldsymbol{\epsilon}| = 1$, and $\mathcal{V}(\mathbf{r}^k) \approx \mathcal{V} \equiv V_0(a^\dagger + a)$].

Superradiant phase transitions and no-go theorem.—In the limit $N \to \infty$, $\mathcal{H}_{\rm D}$ undergoes a second order phase transition at a critical coupling strength [6–8]

$$\lambda_c^2 = \frac{\omega\Omega}{4}\left(1 + \frac{4\kappa}{\omega}\right). \tag{3}$$

This phase transition was discovered for $\mathcal{H}_{\rm D}$ with $\kappa = 0$ and termed SPT [6]; see [9] for recent studies. At λ_c, the atoms polarize spontaneously, $\langle\sum_k \sigma_z^k\rangle/N \neq -1$, and a macroscopic photon occupation arises, $\langle a^\dagger a\rangle/N \neq 0$. A gapless excitation signals the critical point [Fig. 2(a)].

In cavity QED systems, however, λ_c cannot be reached if the κ term is not neglected [8]. That is because λ and κ are not independent of each other. Let us define a parameter α via $\kappa = \alpha\lambda^2/\Omega$. Then Eq. (3) becomes $\lambda_c^2(1 - \alpha) = \omega\Omega/4$, and criticality requires $\alpha < 1$. With $A_0 = 1/\sqrt{2\epsilon_0\omega V}$ (V is the volume of the cavity) one finds

$$\lambda_{\rm cav} = \frac{\Omega|\boldsymbol{\epsilon}\cdot\mathbf{d}|}{\sqrt{2\epsilon_0\omega}}\sqrt{\frac{N}{V}}, \qquad \kappa_{\rm cav} = \frac{n}{2\epsilon_0\omega}\frac{e^2}{2m}\frac{N}{V}, \tag{4}$$

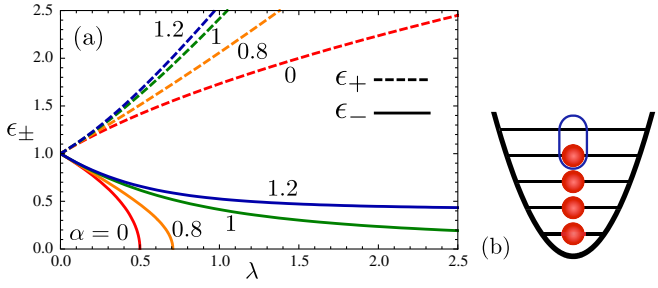

FIG. 2 (color online). (a) Excitation energies ϵ_+ and ϵ_- of the Dicke Hamiltonian $\mathcal{H}_{\rm D}$ versus coupling λ (in units of $\omega = \Omega$), for $\alpha = \kappa\Omega/\lambda^2 = 0, 0.8, 1, 1.2$. For $\alpha = 0$, ϵ_- vanishes at $\lambda = 0.5$, thus signaling a SPT. Only $\alpha \geq 1$ is compatible with the TRK sum rule. For these α, $\epsilon_- \to \sqrt{1 - 1/\alpha}$ and remains finite for all λ. The excitations $\epsilon_\pm(\lambda)$ of $\mathcal{H}_{\rm tm}$ correspond to $\alpha = 1$. (b) Toy model of an (artificial) atom. The oval line indicates the degree of freedom in the simplified effective model.

where $\mathbf{d} = \langle g|e\sum_{i=1}^n \mathbf{r}_i|e\rangle$ and $\alpha_{\rm cav}\Omega|\boldsymbol{\epsilon}\cdot\mathbf{d}|^2 = ne^2/2m$. But the Thomas-Reiche-Kuhn sum rule (TRK) ([16], Sec. A)

$$\sum_l (E_l - E_g)|\boldsymbol{\epsilon}\cdot\langle g|e\sum_{i=1}^n \mathbf{r}_i|l\rangle|^2 = n\frac{e^2}{2m} \tag{5}$$

for the Hamiltonian $H^0 = \sum_{i=1}^n \mathbf{p}_i^2/2m + V_{\rm int}(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ of an uncoupled atom with spectrum $\{E_l, |l\rangle\}$ implies $\Omega|\boldsymbol{\epsilon}\cdot\mathbf{d}|^2 \leq ne^2/2m$, consequently $\alpha_{\rm cav} \geq 1$. This is known as the no-go theorem for SPTs [8,10]. Notice that $\alpha_{\rm cav}$ determines how strongly $\Omega|\boldsymbol{\epsilon}\cdot\mathbf{d}|^2$ exhausts the TRK. We remark that a direct dipole-dipole coupling between atoms (omitted here) can lead to a ferroelectric phase transition, which, however, occurs only at very high atomic densities [17].

Surprisingly, the no-go theorem was recently argued not to apply in circuit QED [10]. Indeed, the standard EM of circuit QED yields

$$\lambda_{\rm cir} = \frac{eC_G}{C_G + C_J}\sqrt{\frac{\omega N}{Lc}}, \qquad \kappa_{\rm cir} = \frac{C_G^2}{2(C_G + C_J)}\frac{\omega N}{Lc}, \tag{6}$$

where L denotes the length of the transmission line resonator, c its capacitance per unit length, and we have used $V_0 = (\omega/Lc)^{1/2}$ [15]. Here $\alpha_{\rm cir} = E_J/4E_C < 1$ is easily possible [1]. According to this argument, a SPT should be observable in a circuit QED system.

Effective models and superradiant phase transitions.—The EM has proved to be a very successful description of circuit QED whose predictions have been confirmed in numerous experiments. However, the circuit QED setups operated so far contained only few artificial atoms. It is not obvious that an EM also provides a good description of circuit QED systems with $N \gg 1$ atoms and, thus, a proper starting point to study SPTs in circuit QED. We now present a toy model illustrating how an EM similar to the one in circuit QED can erroneously predict a SPT.

The toy model consists of N harmonic oscillator potentials with frequency Ω, each trapping n noninteracting fermions of mass m and charge e, which all couple to a bosonic mode with frequency ω [Fig. 2(b)].

This toy model can be viewed as a very simplified description of (artificial) atoms with n microscopic constituents inside a resonator. It is governed by the Hamiltonian

$$\mathcal{H}_{\rm tm} = \omega a^\dagger a + \sum_{k=1}^N\sum_{i=1}^n \frac{(p_i^k - eA)^2}{2m} + \frac{m\Omega^2(x_i^k)^2}{2}, \tag{7}$$

where we assume again $A(x_i^k) \approx A = A_0(a^\dagger + a)$. Since A couples only to the center of mass coordinate of the kth oscillator, $\mathcal{H}_{\rm tm}$ can be diagonalized ([16], Sec. B):

$$\mathcal{H}_{\rm tm} = \epsilon_{\pm}\left(a_{\pm}^{\dagger}a_{\pm} + \frac{1}{2}\right) + \sum_{i=1}^{nN-1}\Omega\left(b_i^{\dagger}b_i + \frac{1}{2}\right),$$

$$2\epsilon_{\pm}^2(\lambda) = \omega^2 + 4\kappa\omega + \Omega^2 \pm \sqrt{(\omega^2 + 4\kappa\omega - \Omega^2)^2 + 16\lambda^2\omega\Omega}. \quad (8)$$

Here, $a_{\pm}^{\dagger}$ generate excitations that mix photon field with collective center of mass motion, the $b_i^{\dagger}$ excite the remaining degrees of freedom, $\lambda = A_0\Omega d\sqrt{N}$ and $\kappa = \lambda^2/\Omega$. As $d = \langle n|ex|n-1\rangle = e\sqrt{n/2m\Omega}$, the TRK is exhausted. Note that $\epsilon_{\pm}(\lambda)$ are also the relevant excitation energies of $\mathcal{H}_{\rm D}$ for $N \to \infty$, as can be shown using methods of Ref. [9] ([16], Sec. B), and demanding $\epsilon_{-} = 0$ yields Eq. (3). One sees that $\epsilon_{\pm}(\lambda)$ is real and nonzero for all λ and that the ground state energy is an analytic function of λ [Fig. 2(a)]. Hence, no phase transition is possible.

Let us now consider an EM for the toy model. Similar to the standard EM of circuit QED, we focus on the fermion with the highest energy in the kth harmonic oscillator and treat it as a two-level system with $|g_k\rangle = |n-1\rangle_k$ and $|e_k\rangle = |n\rangle_k$ [Fig. 2(b)]. Accounting only for one fermion per "atom," that is, expanding $\mathcal{H}_{\rm tm}^{\rm EM} = \omega a^{\dagger}a + \sum_{k=1}^{N}(p^k - eA)^2/2m + m\Omega^2(x^k)^2/2$ in the basis $\{|n-1\rangle_k, |n\rangle_k\}$, yields a Dicke Hamiltonian with $\lambda_{\rm EM} = \lambda$ and $\kappa_{\rm EM} = \kappa/n = Ne^2A_0^2/2m$. Crucially, only $\lambda_{\rm EM}$ depends on n. This allows $\lambda_{\rm EM}$ to be increased at constant $\kappa_{\rm EM}$; therefore, $\alpha_{\rm EM} = 1/n$ can be <1 and a SPT is possible. This failure of the EM can be interpreted as follows. The relation $\lambda = \lambda_{\rm EM} \propto d \propto \sqrt{n}$ reveals that the coupling of an "atom" to the bosonic mode is fully captured by the EM and grows with atom size n. However, in a proper description of the system, increasing the coupling by increasing n unavoidably also increases κ in proportion to n: all fermions of all atoms couple to the bosonic mode and each causes an A^2 term. This is lost in the EM with only 1 degree of freedom per atom. Interestingly, $\alpha_{\rm EM} < 1$ only if $n > 1$, i.e., as long as the effective description actually neglects degrees of freedom.

Microscopic description of circuit QED.—This example suggests not to rely on the standard description for investigating SPTs in circuit QED. Although the dipole coupling of field and qubit states might be fully represented by $\lambda_{\rm cir}$, $\kappa_{\rm cir}$ could still underestimate the A^2 terms of all charged particles in the Cooper-pair boxes. Instead, let us describe a circuit QED system with N artificial atoms by a minimal-coupling Hamiltonian that accounts for all microscopic degrees of freedom:

$$\mathcal{H}_{\rm mic} = \omega a^{\dagger}a + \sum_{k=1}^{N}\sum_{i=1}^{n_k}\frac{(\mathbf{p}_i^k - q_i^k\mathbf{A})^2}{2m_i^k} + V_{\rm int}(\mathbf{r}_1^k, \ldots, \mathbf{r}_{n_k}^k).$$

As we allow arbitrary charges q_i^k and masses m_i^k and an arbitrary interaction potential $V_{\rm int}$ of the n_k constituents of the kth artificial atom, $\mathcal{H}_{\rm mic}$ most generally captures the coupling of N arbitrary (but mutually noninteracting) objects to the electromagnetic field. We subject it to the same approximations that led from $\mathcal{H}_{\rm cir}^0$, the EM of circuit QED, to $\mathcal{H}_{\rm D}$. For identical artificial atoms $\{n_k, q_i^k, m_i^k\} \to \{n, q_i, m_i\}$. The Hamiltonian of an uncoupled artificial atom then reads $H_{\rm mic}^0 = \sum_{i=1}^{n}\mathbf{p}_i^2/2m_i + V_{\rm int}(\mathbf{r}_1, \ldots, \mathbf{r}_n)$. Its qubit states $|g\rangle$ and $|e\rangle$, which in the standard EM are superpositions of the charge states $|\nu\rangle$, are among the eigenstates $\{|l\rangle\}$ of $H_{\rm mic}^0$. Expanding $\mathcal{H}_{\rm mic}$ in the $\{|g\rangle_k, |e\rangle_k\}$ basis and taking $\mathbf{A}(\mathbf{r}_i^k) \approx \mathbf{A}$ gives the Dicke Hamiltonian $\mathcal{H}_{\rm D}$ with parameters generalizing those of cavity QED [Eq. (4)],

$$\lambda_{\rm cir}^{\rm mic} = \frac{\Omega|\boldsymbol{\epsilon}\cdot\mathbf{d}|}{\sqrt{2\epsilon_0\omega}}\sqrt{\frac{N}{V}}, \qquad \kappa_{\rm cir}^{\rm mic} = \frac{1}{2\epsilon_0\omega}\left(\sum_{i=1}^{n}\frac{q_i^2}{2m_i}\right)\frac{N}{V}, \quad (9)$$

where $\mathbf{d} = \langle g|\sum_{i=1}^{n} q_i\mathbf{r}_i|e\rangle$. This microscopic description of circuit QED facilitates the same line of argument which in Ref. [8] allowed the conclusion that there is no SPT in cavity QED: Criticality [Eq. (3)] requires $\Omega|\boldsymbol{\epsilon}\cdot\mathbf{d}|^2 > \sum_{i=1}^{n} q_i^2/2m_i$, which is ruled out by TRK for $H_{\rm mic}^0$,

$$\sum_l (E_l - E_g)|\boldsymbol{\epsilon}\cdot\langle g|\sum_{i=1}^{n} q_i\mathbf{r}_i|l\rangle|^2 = \sum_{i=1}^{n}\frac{q_i^2}{2m_i}. \quad (10)$$

Hence, the no-go theorem of cavity QED applies to circuit QED as well. This result confirms the analogy of cavity and circuit QED also with respect to SPTs. It has been obtained under the same approximations that led from the standard description of circuit QED, $\mathcal{H}_{\rm cir}^0$, to $\mathcal{H}_{\rm D}$ with $\lambda_{\rm cir}$ and $\kappa_{\rm cir}$. The discrepancy of the predictions of the microscopic and the standard description of circuit QED thus shows the limitations of the validity of the latter. This might be important for future circuit QED architectures with many artificial atoms in general, even for applications not related to SPTs. We emphasize, though, that our conclusion neither forbids SPTs in circuit QED systems with inductively coupling flux qubits [18] nor is it at odds with the great success of the standard description for few-atom systems: there, the deficiency of $\kappa_{\rm cir}$ does not manifest itself qualitatively as the κ term in $\mathcal{H}_{\rm D}$ mimics slightly renormalized system parameters $\tilde{\omega}$ and $\tilde{\lambda}$.

Possible loophole in the no-go theorem.—Although the two-level approximation for the anharmonic spectrum of (artificial) atoms is well justified in many cases, one might argue that higher levels should be taken into account in this context. Indeed, a SPT does not require $\Omega \approx \omega$, and thereby does not single out a particular atomic transition.

For a more profound reason for dropping the two-level assumption, consider the elementary question of how the presence of N mutually noninteracting atoms shifts a resonator's frequency ω. This situation is described by $\mathcal{H}_{\rm mic}$. It can be rewritten as $\mathcal{H}_{\rm mic} = \omega a^{\dagger}a + \sum_{k=1}^{N}(H_{\rm mic}^k + \mathcal{H}_{pA}^k + \mathcal{H}_{A^2}^k)$, where $\mathcal{H}_{pA}^k$ and $\mathcal{H}_{A^2}^k$ are the $\mathbf{pA}$ and $\mathbf{A}^2$ terms due to the kth atom ([16], Sec. C). Let us perturbatively calculate the frequency shift $\delta\omega = \delta\omega_{pA} + \delta\omega_{A^2}$ caused by $\sum\mathcal{H}_{pA}^k$ and $\sum\mathcal{H}_{A^2}^k$ ([16], Sec. C). To this end,

take $\omega \ll \Omega_m^k$ for all m, k, where Ω_m^k is the mth excitation energy of $H_{\rm mic}^k$. Remarkably, it turns out ([16], Sec. C) that $\delta\omega_{pA}$ (<0) and $\delta\omega_{A^2}$ (>0) cancel almost exactly due to the TRK. The total frequency shift is small, $\delta\omega \sim (\omega/\Omega_m^k)^2$. As a SPT equates to $\delta\omega = -\omega$, the significance of both **pA** and $\mathbf{A}^2$ terms for its existence becomes clear. The **pA** terms cause a strong negative shift and favor a SPT, the $\mathbf{A}^2$ terms do the opposite. This means, most crucially, that one must not unequally truncate **pA** and $\mathbf{A}^2$ terms for assessing the possibility of a SPT by an approximate Hamiltonian. Dropping the $\mathbf{A}^2$ terms in $\mathcal{H}_{\rm D}$ ($\kappa = 0$) leads to the prediction of a SPT. In contrast, $\mathcal{H}_{\rm D}$ with $\kappa \neq 0$ fully incorporates the $\mathbf{A}^2$ terms of $\mathcal{H}_{\rm mic}$. But, due to the two-level approximation, it has only one matrix element of the **pA** terms per atom, thereby possibly underestimating the tendency towards a SPT. To exclude SPTs in cavity and circuit QED, a generalization of the no-go theorem to (artificial) atoms with more than two energy levels is necessary.

Generalized no-go theorem.—Let us consider $N \to \infty$ identical atoms coupled to a field mode with frequency ω. The atomic Hamiltonians $H_{\rm mic}^k$ may have an arbitrary spectrum $\{\Omega_l, |l_k\rangle = |l\rangle_k\}$, with $\Omega_0 = 0$ and μ excited states (Fig. 3).

With $d_{l,l'} = \boldsymbol{\epsilon} \cdot \langle l| \sum_{i=1}^n q_i \mathbf{r}_i |l'\rangle$, the full Hamiltonian of the system reads

$$\mathcal{H}_{\rm mic} = \omega a^\dagger a + \kappa(a^\dagger + a)^2 + \sum_{k=1}^{N} \sum_{l,l'=0}^{\mu} (\Omega_l \delta_{l,l'} |l_k\rangle\langle l_k| + iA_0(\Omega_{l'} - \Omega_l) d_{l,l'} (a^\dagger + a) |l_k\rangle\langle l'_k|). \quad (11)$$

We now follow a strategy similar to that of Refs. [9]: We derive a generalized Dicke Hamiltonian $\mathcal{H}_{\rm GD}$ having the same low-energy spectrum as $\mathcal{H}_{\rm mic}$ for a small density of atoms, $N/V \simeq 0$, using $A_0 \propto V^{-1/2}$ as small parameter. We then check whether $\mathcal{H}_{\rm GD}$ has a gapless excitation if the density is increased, which would signal a SPT and mark the breakdown of the analogy of $\mathcal{H}_{\rm GD}$ and $\mathcal{H}_{\rm mic}$.

Expanding the eigenstates and eigenenergies of $\mathcal{H}_{\rm mic}$ as $|\mathcal{E}\rangle \propto \sum_{s=0}^{\infty} A_0^s |\mathcal{E}_s\rangle$ and $\mathcal{E} \propto \sum_{ss'} A_0^{s+s'} \langle \mathcal{E}_s | \mathcal{H}_{\rm mic} | \mathcal{E}_{s'} \rangle$, we note that contributions from all $d_{l\neq 0, l'\neq 0}$ terms may be neglected: they are smaller than those retained by a factor of at least one power of A_0 (for $s + s' > 1$) or $\xi/N \ll 1$ (for $s + s' \leq 1$), where $\xi = \sum_k \sum_{l>0} |\langle l_k | \mathcal{E}_0 \rangle|^2$ is the number of atomic excitations in $|\mathcal{E}_0\rangle$, which is $\ll N$ for low-lying eigenstates ([16], Sec. D). We thus define $\mathcal{H}_{\rm GD}$ by setting $d_{l\neq 0, l'\neq 0} \to 0$ in $\mathcal{H}_{\rm mic}$. Up to a constant, we find ([16], Sec. D)

$$\mathcal{H}_{\rm GD} = \tilde{\omega} a^\dagger a + \sum_{l=1}^{\mu} \Omega_l b_l^\dagger b_l + \sum_{l=1}^{\mu} \tilde{\lambda}_l (b_l^\dagger + b_l)(a^\dagger + a), \quad (12)$$

by introducing $b_l^\dagger = \frac{1}{\sqrt{N}} \sum_{k=1}^N |l_k\rangle\langle 0_k|$ as collective excitation, omitting the energy of the "dark" collective excitations ([16], Sec. D), and removing the κ term by a Bogolyubov transformation yielding $\omega \to \tilde{\omega} = \sqrt{\omega^2 + 4\kappa\omega}$ and $\lambda_l \to \tilde{\lambda}_l = \sqrt{\frac{\omega}{\tilde{\omega}}} \lambda_l$, with $\lambda_l = A_0 \Omega_l |d_{0,l}| \sqrt{N}$. For dilute excitations, the b_l are bosonic, $[b_l, b_{l'}^\dagger] = \delta_{l,l'}$ [19]. The system undergoes a SPT if an eigenfrequency ϵ_i of $\mathcal{H}_{\rm GD}$ can be pushed to zero by increasing the couplings λ_l. We cannot calculate the ϵ_i's explicitly, but we will show that the assumption $\epsilon_i = 0$ contradicts the TRK. An ϵ_i solves the characteristic equation ([16], Sec. D)

$$\left(\prod_{l'=1}^{\mu} (\Omega_{l'}^2 - \epsilon^2) \right) \left((\tilde{\omega}^2 - \epsilon^2) - 4\tilde{\omega} \sum_{l=1}^{\mu} \frac{\Omega_l \tilde{\lambda}_l^2}{\Omega_l^2 - \epsilon^2} \right) = 0. \quad (13)$$

If ϵ_i were zero, this would imply

$$\frac{\omega}{4NA_0^2} = \sum_{l=1}^{\mu} \Omega_l |d_{0l}|^2 - \sum_{i=1}^{n} \frac{q_i^2}{2m_i} \quad (14)$$

and contradict the TRK for $H_{\rm mic}^0$ [Eq. (10)], which ensures that the right-hand side is negative even if the entire atomic spectrum is incorporated. This result is irrespective of the details of the atomic spectra. Note that for $\kappa = 0$, the negative term on the right-hand side of Eq. (14) vanishes, and one recovers the SPT for critical couplings λ_{lc} with $\sum_{l=1}^{\mu} \lambda_{lc}^2/\Omega_l = \omega/4$. This resembles Eq. (3) with $\kappa = 0$.

Experimental evidence for our conclusions could be gained by probing the shifted resonator frequency of a suitable circuit QED system. Consider a sample containing N artificial atoms with $\lambda/\sqrt{N} = 2\pi \times 120$ MHz and $\Omega/2\pi = \omega/2\pi = 3$ GHz. If $\alpha_{\rm cir} = E_J/4E_C = 0.1$, as predicted by the standard theory, there should be signatures of criticality for $N = 174$ [according to Eq. (3)], and the resonator frequency should be close to zero. But even if we assume $\alpha = 1$, the minimal value compatible with the TRK (that corresponds to ideal two-level atoms), we find the lowest excitation ϵ_- to be still at $\epsilon_- \approx 2\pi \times 2$ GHz. We have verified that these phenomena are insensitive to small fluctuations of the atomic parameters ([16], Sec. E; see also [18]) and hence experimentally observable.

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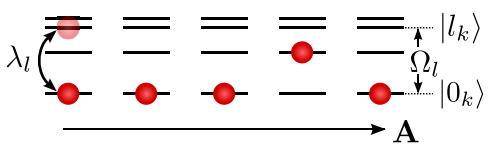

FIG. 3 (color online). Situation of the generalized no-go theorem. Many multilevel (artificial) atoms couple to the photon field. Transitions between excited atomic states are irrelevant for the low-energy spectrum of the system.

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