Holographic Dual of a Boundary Conformal Field Theory

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We propose a holographic dual of a conformal field theory defined on a manifold with boundaries, i.e., boundary conformal field theory (BCFT). Our new holography, which may be called anti–de Sitter BCFT, successfully calculates the boundary entropy or g function in two-dimensional BCFTs and it agrees with the finite part of the holographic entanglement entropy. Moreover, we can naturally derive a holographic g theorem. We also analyze the holographic dual of an interval at finite temperature and show that there is a first order phase transition.

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The anti–de Sitter (AdS) conformal field theory (CFT) correspondence is a very fascinating idea which enables us to study quantum gravity in a nonperturbative way and at the same time to analyze strongly coupled CFTs efficiently [1,2]. The purpose of this Letter is to consider the holographic dual of CFT defined on a manifold M with a boundary ∂M, which is the so called boundary conformal field theory (BCFT). We argue that this is given by generalizing the AdS CFT correspondence in the following way. Based on the idea of holography [3], we extend a d-dimensional manifold M to a $d+1$ dimensional asymptotically AdS space N so that $\partial N = M \cup Q$, where Q is a d-dimensional manifold which satisfies $\partial Q = \partial M$. See Fig. 1 for some examples of our construction.

Usually, we impose the Dirichlet boundary condition on the metric at the AdS boundary. Thus we assume the Dirichlet boundary condition on M. However, we propose to require a Neumann boundary condition on the metric at Q as explained later. This can be regarded as a modification of the well-known Randall-Sundrum setup [4] (see also, e.g., [5] in the context of AdS CFT) such that the brane now intersects with the AdS boundary. Refer also to [6], where microscopic descriptions in string theory for a variety of boundary conditions have been discussed. An earlier paper [7] also presents another way to introduce a boundary in AdS CFT.

To make the variational problem sensible, we usually add the Gibbons-Hawking boundary term [8] to the Einstein-Hilbert action (we omit the boundary term for M):

$$I = \frac{1}{16\pi G_N}\int_N \sqrt{-g}(R-2\Lambda) + \frac{1}{8\pi G_N}\int_Q \sqrt{-h}K. \quad (1)$$

The metric of N and Q is denoted by g and h, respectively. $K = h^{ab}K_{ab}$ is the trace of extrinsic curvature K_{ab} defined by $K_{ab} = \nabla_a n_b$, where n is the unit vector normal to Q with a projection of indices onto Q from N.

Consider the variation of the metric in the above action. After a partial integration, we find

$$\delta I = \frac{1}{16\pi G_N}\int_Q \sqrt{-h}(K_{ab}\delta h^{ab} - Kh_{ab}\delta h^{ab}). \quad (2)$$

Notice that the terms which involve the derivative of δh_{ab} cancel out thanks to the boundary term. We can add to (1) the action I_Q of some matter fields localized on Q. This prescription phenomenologically provides holographic duals of various boundary conditions in the CFT. We impose the Neumann boundary condition instead of the Dirichlet one by setting the coefficients of δh^{ab} to zero and finally we obtain the boundary condition

$$K_{ab} - h_{ab}K = 8\pi G_N T^Q_{ab}, \quad (3)$$

where we defined $T^{Qab} = (2/\sqrt{-h})\delta I_Q/\delta h_{ab}$.

As a simple example we would like to assume that the boundary matter Lagrangian is just a constant. This leads us to consider the following action:

$$I = \frac{1}{16\pi G_N}\int_N \sqrt{-g}(R-2\Lambda) + \frac{1}{8\pi G_N}\int_Q \sqrt{-h}(K-T). \quad (4)$$

The constant T is interpreted as the tension of the boundary surface Q. In AdS CFT, a $d+1$ dimensional AdS space (AdSd_{+1}) is dual to a d-dimensional CFT. The geometrical $SO(2,d)$ symmetry of AdS is equivalent to the conformal symmetry of the CFT. When we put a $d-1$ dimensional boundary to a d-dimensional CFT such that the presence of the boundary breaks $SO(2,d)$ into $SO(2,d-1)$, this is

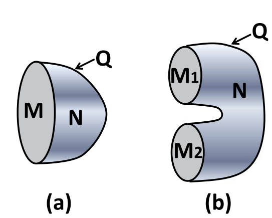

FIG. 1 (color online). Examples of the holographic duals of BCFT with a single AdS boundary (a) and two AdS boundaries (b).

called a BCFT [9]. Note that though the holographic duals of defect or interface CFTs [10,11] look very similar with respect to the symmetries, their gravity duals are different from ours because they do not have extra boundaries like Q.

To realize this structure of symmetries, we take the following ansatz of the metric (see also [10,12]):

$$ds^2 = d\rho^2 + \cosh^2\frac{\rho}{R} ds^2_{\mathrm{AdS}_d}. \tag{5}$$

If we assume that ρ takes all values from $-\infty$ to ∞, then (5) is equivalent to the AdS_{d+1}. To see this, let us assume the Poincaré metric of AdS_d by setting

$$ds^2_{\mathrm{AdS}_d} = R^2\frac{-dt^2 + dy^2 + d\vec{w}^2}{y^2}, \tag{6}$$

where $\vec{w} \in R^{d-2}$. Remember that the cosmological constant Λ is related to the AdS radius R by $\Lambda = -\frac{d(d-1)}{2R^2}$.

By defining new coordinates z and x as $z = y/\cosh\frac{\rho}{R}$, $x = y\tanh\frac{\rho}{R}$, we recover the familiar form of the Poincaré metric of AdS_{d+1}: $ds^2 = R^2(dz^2 - dt^2 + dx^2 + d\vec{w}^2)/z^2$.

To realize a gravity dual of BCFT, we will put the boundary Q at $\rho = \rho_*$ and this means that we restrict the spacetime to the region $-\infty < \rho < \rho_*$ [as described in Fig. 2(a)]. The extrinsic curvature on Q reads $K_{ab} = (1/R)\tanh(\rho/R)h_{ab}$. The boundary condition (3) leads to

$$K_{ab} = (K - T)h_{ab}. \tag{7}$$

Thus ρ_* is determined by the tension T as follows:

$$T = \frac{d-1}{R}\tanh\frac{\rho_*}{R}. \tag{8}$$

Let us concentrate on the $d = 2$ case to describe the two-dimensional BCFT. This setup is special in that it has been well studied (see [13] and references therein) and that the BCFT has an interesting quantity called the boundary entropy (or g function) [14]. The boundary state of a BCFT with a boundary condition α is denoted by $|B_\alpha\rangle$ below. We define the quantity called g by the disk amplitude $g_\alpha = \langle 0|B_\alpha\rangle$, where $|0\rangle$ is the vacuum state. The boundary entropy $S^{(\alpha)}_{\mathrm{bdy}}$ is defined by

$$S^{(\alpha)}_{\mathrm{bdy}} = \log g_\alpha. \tag{9}$$

The boundary entropy measures the boundary degrees of freedom and can be regarded as a boundary analogue of the central charge c.

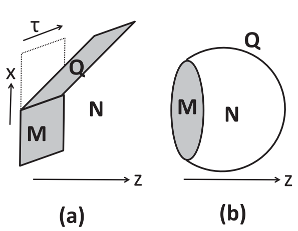

FIG. 2. The holographic dual of a half line (a) and a disk (b).

Consider a holographic dual of a CFT on a round disk defined by $\tau^2 + x^2 \le r_D^2$ in the Euclidean AdS_3 spacetime

$$ds^2 = R^2\frac{dz^2 + d\tau^2 + dx^2}{z^2}, \tag{10}$$

where τ is the Euclidean time. In the Euclidean formulation, the action (4) is now replaced by

$$I_E = -\frac{1}{16\pi G_N}\int_N \sqrt{g}(R - 2\Lambda) - \frac{1}{8\pi G_N}\int_Q \sqrt{h}(K - T). \tag{11}$$

Note that ρ_* is related to the tension T of the boundary via (8). When the BCFT is defined on the half-space $x < 0$, its gravity dual has been found in the previous section. Therefore we can find the gravity dual of the BCFT on the round disk by applying the conformal map (see, e.g., [15]). The final answer is the following domain in AdS_3:

$$\tau^2 + x^2 + [z - \sinh(\rho_*/R)r_D]^2 - r_D^2\cosh^2(\rho_*/R) \le 0. \tag{12}$$

In this way we found that the holographic dual of BCFT on a round disk is given by a part of the two-dimensional round sphere [see Fig. 2(b)]. A larger value of tension corresponds to the larger radius.

Now we would like to calculate the disk partition function in order to obtain the boundary entropy. By evaluating (11) in the domain (12), we obtain

$$I_E = \frac{R}{4G_N}\left[\frac{r_D^2}{2\epsilon^2} + \frac{r_D\sinh(\rho_*/R)}{\epsilon} + \log(\epsilon/r_D) - \frac{1}{2} - \frac{\rho_*}{R}\right], \tag{13}$$

where we introduced the UV cutoff $z > \epsilon$ as usual. By adding the counterterm on the AdS boundary [16], we can subtract the divergent terms in (13). The difference of the partition function between $\rho = 0$ and $\rho = \rho_*$ is given by $I_E(\rho_*) - I_E(0) = -\frac{\rho_*}{4G_N}$. Since the partition function is given by $Z = e^{-S_E}$, we obtain the boundary entropy

$$S_{\mathrm{bdy}} = \frac{\rho_*}{4G_N}, \tag{14}$$

where we assumed $S_{\mathrm{bdy}} = 0$ for $T = 0$ because the boundary contributions vanish in this case.

Another way to extract the boundary entropy is to calculate the entanglement entropy. The entanglement entropy S_A with respect to the subsystem A is defined by the von Neumann entropy $S_A = -\mathrm{Tr}\rho_A\log\rho_A$ for the reduced density matrix ρ_A. In a two-dimensional CFT on a half line, S_A behaves as follows [17]:

$$S_A = \frac{c}{6}\log\frac{l}{\epsilon} + \log g, \tag{15}$$

where c is the central charge and ϵ is the UV cutoff (or lattice spacing); A is chosen to be an interval with length l

such that it ends at the boundary. The $\log g$ in (15) coincides with the boundary entropy (9).

In AdS CFT, the holographic entanglement entropy is given in terms of the area of the codimension two minimal surface (called γ_A) which ends at ∂A [18] $S_A = \text{Area}(\gamma_A)/4G_N$. Using this formula, the boundary entropy in interface CFTs has successfully been calculated in [12,19].

Consider the gravity dual of a two-dimensional BCFT on a half line $x < 0$ in the coordinate (10). By taking the time slice $\tau = 0$, we define the subsystem A by the interval $-l \le x \le 0$. In this case, the minimal surface (or geodesic line) γ_A is given by $x^2 + z^2 = L^2$. If we go back to the coordinate system (5) and (6), then γ_A is simply given by $\tau = 0$, $y = l$ and $-\infty < \rho \le \rho_*$. This leads to $S_A = (1/4G_N)\int_{-\infty}^{\rho_*} d\rho$. By subtracting the bulk contribution which is divergent as in (15), we reproduce the previous result (14).

In two-dimensional CFT, there is a well-known fact, the so called c theorem [20], that the central charge monotonically decreases under the renormalization-group (RG) flow. In the case of BCFT, an analogous quantity is actually known to be the g function or, equally, boundary entropy [14]. At fixed points of boundary RG flows, it is reduced to that of BCFT introduced in (9). It has been conjectured that the g function monotonically decreases under the boundary RG flow in [14] and this has been proven in [21] later. Therefore the holographic proof of g theorem described below will offer us important evidence of our proposed holography. Refer to [22] for a holographic c theorem and to [23] for a holographic g theorem in the defect CFT under a probe approximation.

Because we want to keep the bulk conformal invariance and we know that all solutions to the vacuum Einstein equation with $\Lambda < 0$ are locally AdS_3, we expect that the bulk spacetime remains to be AdS_3 as long as matter fields are not excited. We describe the boundary Q by the curve $x = x(z)$ in the metric (10). We assume generic matter fields on Q and this leads to the energy stress tensor T^Q_{ab} term in the boundary condition (3). It is easy to check the energy conservation $\nabla^a T^Q_{ab} = 0$ in our setup because $\nabla^a(K_{ab} - Kh_{ab}) = R_{nb}$, where n is the Gaussian normal coordinate which is normal to Q. In order to require that the matter fields on the boundary are physically sensible, we impose the null energy condition (or weaker energy condition) as in the holographic c theorem [22]. It is given by the inequality $T^Q_{ab}N^aN^b \ge 0$ for any null vector N^a tangent to the surface Q. This condition is equivalent to $x''(z) \le 0$ because

$$(K_{ab} - Kh_{ab})N^aN^b = -\frac{Rx''(z)}{z[1 + x'(z)^2]^{3/2}} \ge 0, \tag{16}$$

for the two null vectors $(N^t, N^z, N^x) = [\pm 1, (1 + x'^2)^{-1/2}, x'(1 + x'^2)^{-1/2}]$. Since at a fixed point the boundary entropy is given by $S_{\text{bdy}} = \frac{\rho_*}{4G_N}$ and we have the relation $\frac{x}{z} = \sinh(\rho_*/R)$ on the boundary Q, we would like to propose the following g function $\log g(z) = \frac{R}{4G_N}\text{arcsinh}(\frac{x(z)}{z})$. By taking the derivative, we get $\partial \log g(z)/\partial z = \frac{x'(z)z - x(z)}{\sqrt{z^2 + x(z)^2}}$. Indeed we can see that $x'z - x$ is nonpositive because this is vanishing at $z = 0$ and (16) leads to $(x'z - x)' = x''z \le 0$. In this way, we manage to derive the g theorem.

Since so far we have studied a holographic BCFT in the presence of a single boundary, next we would like to analyze a holographic dual of a two-dimensional CFT on an interval. At finite temperature, there are two candidates for the bulk geometry: one of them is the thermal AdS_3 and the other is the Banados, Teitelboim, and Zanelli (BTZ) black hole (AdS_3 black hole). In the absence of boundaries Q, there is the well-known Hawking-Page phase transition between them [24,25].

At low temperature, the bulk geometry is expected to be given by the thermal AdS_3 defined by the metric

$$ds^2 = R^2\frac{d\tau^2}{z^2} + R^2\frac{dz^2}{h(z)z^2} + \frac{R^2h(z)}{z^2}dx^2, \tag{17}$$

where $h(z) = 1 - (z/z_0)^2$. The periodicity of the Euclidean time τ, denoted by the inverse temperature $1/T_{\text{BCFT}}(\equiv 2\pi z_H)$, can be chosen arbitrarily, while that of the space direction x is determined to be $2\pi z_0$ by requiring smoothness.

We again describe the boundary Q by the curve $x = x(z)$. The boundary condition (7) is solved as follows:

$$x(z) - x(0) = z_0 \arctan\left(\frac{RTz}{z_0\sqrt{h(z) - R^2T^2}}\right). \tag{18}$$

Notice that $x'(z)$ gets divergent at $z_* = z_0\sqrt{1 - R^2T^2}$ and thus this should be the turning point [see Fig. 3(a)]. Thus the boundary Q totally extends from $x = 0$ to $x = \pi z_0$. In the end, we can evaluate the action

$$I_E = -\frac{\pi R z_H}{8G_N z_0} = -\frac{\pi}{24}\frac{c}{\Delta x T_{\text{BCFT}}}, \tag{19}$$

where we employed the well-known relation between the AdS_3 radius R and the central charge c of CFT_2, given by $c = \frac{3R}{2G_N}$ [26]. Note that the final result (19) does not depend on the tension T and is correct even when $T < 0$.

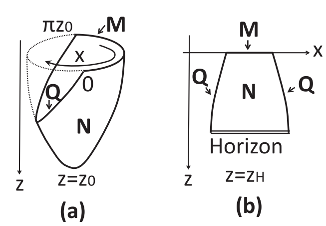

FIG. 3. The holographic dual of an interval at low temperature (a) and high temperature (b).

On the other hand in the higher temperature phase, the bulk is described by a part of the BTZ black hole:

$$ds^2 = R^2 \frac{f(z)}{z^2} d\tau^2 + R^2 \frac{dz^2}{f(z)z^2} + R^2 \frac{dx^2}{z^2}, \tag{20}$$

where $f(z) = 1 - (z/z_H)^2$. The Euclidean time τ is compactified on a circle such that $\tau \sim \tau + 2\pi z_H$ and thus the temperature in the dual BCFT is $T_{\rm BCFT} = \frac{1}{2\pi z_H}$. The length of the interval is again denoted by $\Delta x = \pi z_0$.

We find the following profile $x = x(z)$ of Q from (7)

$$x(z) - x(0) = z_H \mathrm{arcsinh}\left(\frac{RTz}{z_H\sqrt{1 - R^2T^2}}\right). \tag{21}$$

Note Q consists of two disconnected parts as in Fig. 3(b).

Now we evaluate the Euclidean action (11) in the form $I_E = 2I_{\rm bdy} + I_{\rm bulk}$. $2I_{\rm bdy}$ is the boundary contributions, while $I_{\rm bulk}$ is the bulk ones which do not depend on T. After subtracting the divergences, we obtain

$$I_{\rm bulk} = -\frac{\pi c}{6} \Delta x T_{\rm BCFT}. \tag{22}$$

This result (22) clearly agrees with what we expect from the standard CFT results. On the other hand, each of two boundary contributions is found to be

$$I_{\rm bdy} = -\frac{\rho_*}{4G_N} = -\frac{c}{6} \mathrm{arctanh}(RT). \tag{23}$$

This offers us one more independent calculation of boundary entropy via the identity $g = e^{S_{\rm bdy}} = e^{-I_{\rm bdy}}$.

Let us examine when either of the two phases is favored. To see this we compare (22) + 2 × (23) with (19) and pick the smaller one. In this way we find that the black hole phase is realized when

$$\Delta x T_{\rm BCFT} > -\frac{1}{\pi} \mathrm{arctanh}(RT) + \sqrt{\frac{1}{4} + \frac{1}{\pi^2} \mathrm{arctanh}^2(RT)}.$$

At lower temperature, the thermal AdS phase is favored. At vanishing tension $T = 0$, the phase boundary $z_0 = z_H$ coincides with that of the Hawking-Page transition [25]. As the tension gets larger, the critical temperature gets lower. This phase transition is first order and is analogous to the confinement or deconfinement transition in gauge theories [25].

Finally, it is also interesting to consider the case where the boundary M consists of two disconnected manifolds M_A and M_B as in Fig. 1(b). The holographic entanglement entropy S_A between M_A and M_B is estimated as the minimal area of the cross section of the throat [27], which is finite and nonvanishing. Many things are left for future work such as the study of correlation functions, higher dimensional and supersymmetric examples, string or M theory realizations, and applications to condensed matter physics.

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