Anisotropic $\mathcal{N} = 4$ Super-Yang-Mills Plasma and Its Instabilities

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We present a type-IIB supergravity solution dual to a spatially anisotropic finite-temperature $\mathcal{N} = 4$ super-Yang-Mills plasma. The solution is static and completely regular. The full geometry can be viewed as a renormalization group flow from an ultraviolet anti–de Sitter geometry to an infrared Lifshitz-like geometry. The anisotropy can be equivalently understood as resulting from a position-dependent θ term or from a nonzero number density of dissolved $D7$-branes. The holographic stress tensor is conserved and anisotropic. The presence of a conformal anomaly plays an important role in the thermodynamics. The phase diagram exhibits homogeneous and inhomogeneous (i.e., mixed) phases. In some regions the homogeneous phase displays instabilities reminiscent of those of weakly coupled plasmas. We comment on similarities with QCD at finite baryon density and with the phenomenon of cavitation.

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Introduction.—The realization that the quark-gluon plasma (QGP) produced in heavy ion collisions (HIC) is strongly coupled [1] has provided motivation for understanding the dynamics of strongly coupled non-Abelian plasmas through the gauge-string duality [2] (see [3] for a review of applications to the QGP). The simplest example of the duality is the equivalence between four-dimensional $\mathcal{N} = 4$ $SU(N_c)$ super-Yang-Mills (SYM) theory and IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$, where AdS stands for anti–de Sitter space. Here we extend this example to the case in which the SYM plasma is spatially anisotropic. For accessibility by a broad audience, details will appear elsewhere [4]. Previous holographic studies of anisotropic plasmas include [5,6]. One important difference with the gravity solution of [6] is that the latter possesses a naked singularity, whereas our solution is completely regular.

Part of our motivation comes from the fact that the QGP created in HIC is anisotropic. An intrinsically anisotropic hydrodynamic description has been proposed to describe the early stage after the collision [7]. After that stage each little cube of QGP is isotropic in its own rest frame, but even in this phase certain observables may be sensitive to the physics in several adjacent cubes.

Weakly coupled plasmas, both Abelian and non-Abelian, are known to suffer from instabilities in the presence of anisotropies [8]. It is therefore interesting to understand whether this also happens in strongly coupled anisotropic plasmas. Our gravity solution exhibits instabilities reminiscent of weak-coupling instabilities.

At a more theoretical level, motivation is provided by a connection with the fluid-gravity correspondence [9] and the blackfold approach to black hole dynamics [10], both of which assert that the effective theory describing the long-wavelength dynamics of a black hole horizon is a hydrodynamic theory. Inclusion of conserved p-form charges on the gravity side leads to anisotropic hydrodynamics [11].

The IIB supergravity solution that we will present is a finite-temperature generalization of that of [12] and (i) it is static and anisotropic, (ii) it possesses a horizon and it is regular on and outside the horizon, and (iii) it obeys $\mathrm{AdS}_5 \times \mathrm{S}^5$ asymptotic boundary conditions. Staticity is required for simplicity, since, e.g., we would like to study the thermodynamics of the system. The presence of a horizon is dual to the existence of a finite-temperature plasma in the gauge theory. Regularity guarantees that calculations are unambiguous and well defined. The boundary conditions ensure that holography is on its firmest footing and that the solution is solidly embedded in string theory.

As in [12], we deform the SYM theory by a θ parameter that depends linearly on one of the three spatial coordinates, $\theta = 2\pi n_{D7} z$, where n_{D7} is a constant with dimensions of energy. In other words, we add to the SYM action a term $\delta \mathrm{S} \propto \int \theta(z) \mathrm{Tr} F \wedge F$. The system we are describing is therefore a static plasma in thermal equilibrium in the presence of an anisotropic external source. Yet, translation invariance is preserved, since integration by parts yields $\delta \mathrm{S} \propto -n_{D7} \int dz \wedge \mathrm{Tr}(A \wedge F + \frac{2}{3} A^3)$.

The dual gravity description is as follows. Since the θ parameter is dual to the IIB axion χ [13], we expect that in the gravity solution this will be of the form $\chi = az$. It turns out [4] that $a = \lambda n_{D7}/4\pi N_c$, where $\lambda = g_{\mathrm{YM}}^2 N_c$ is the 't Hooft coupling. Since the axion is magnetically sourced by $D7$-branes, the solution can be interpreted in

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terms of a number of $D7$-branes wrapped on the S^5, extending along the xy directions and distributed along the z direction with density $n_{D7} = dN_{D7}/dz$ [4,12]. For this reason we will refer to a and/or n_{D7} as a "charge density." Thus in the gravity description it is clear that isotropy is broken by the presence of anisotropic extended objects. Since their full backreaction is incorporated, the $D7$-branes are completely "dissolved" in the geometry, just like the N_c $D3$-branes that give rise to $AdS_5 \times S^5$. Unlike the case of flavor $D7$-branes [14], the $D7$-branes considered here do not extend in the radial direction. Consequently, they do not reach the AdS boundary and they do not add new degrees of freedom to the SYM theory.

As in [12], the solution can be viewed as a renormalization group (RG) flow between an AdS geometry in the ultraviolet and a Lifshitz-like geometry in the infrared. At $T = 0$ the Lifshitz metric (in string frame) possesses a naked curvature singularity [4], but this is hidden behind the horizon at $T > 0$.

Solution.—The ten-dimensional solution is a direct product, one of whose factors is an S^5 of constant radius L in the Einstein frame. Therefore it can be viewed as a solution of five-dimensional supergravity with cosmological constant $\Lambda = -6/L^2$. Since only the metric g, the axion χ, and the dilaton ϕ are excited, it suffices to consider the axion-dilaton-gravity action

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\left(R + 12 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\right) + S_{GH}, \tag{1}$$

where we have set $L = 1$ and S_{GH} is the Gibbons-Hawking boundary term. The Einstein-frame metric is

$$ds^2 = \frac{e^{-(1/2)\phi}}{u^2}\left(-\mathcal{F}\mathcal{B}dt^2 + dx^2 + dy^2 + \mathcal{H}dz^2 + \frac{du^2}{\mathcal{F}}\right), \tag{2}$$

and $\chi = az$. Isotropy in the xy directions is clearly respected, but not in the z direction unless $\mathcal{H} = 1$. The axion induces the anisotropy. $\mathcal{F}$ is a "blackening factor" that vanishes at the horizon, $u = u_H$. The boundary is at $u = 0$. The dilaton only depends on the radial coordinate u, as do $\mathcal{F}$, $\mathcal{B}$, and $\mathcal{H}$, which are completely determined in terms of ϕ. This in turn obeys a third-order ordinary differential equation which we solved numerically [4]. The temperature is determined from the requirement that the Euclidean continuation of (2) be regular, and the entropy density from the area of the horizon. These quantities are well defined since the solution is static; i.e., the dual plasma is in thermal equilibrium (see the section "Phase diagram"). Figure 1 (left) shows the entropy density as a function of a/T, normalized by the isotropic value $s^0(T) = \pi^2 N_c^2 T^3/2$ [15]. This provides us with the following check. We see from the log-log plot that for small a/T the points lie on the horizontal axis, while for large a/T they lie along a line with slope $1/3$. Thus at $T \gg a$ we recover the isotropic result, whereas at $T \ll a$ we recover the Lifschitz scaling $s \propto a^{1/3}T^{8/3}$ found in [12]. This interpolating behavior is expected from the interpretation of the solution as an RG flow.

Holographic stress tensor.—The energy density and the pressures can be obtained from the holographic stress tensor, whose calculation requires the addition of counterterms to (1). These can be obtained from [16,17] and (in Euclidean signature) take the form

$$S_{ct} = \frac{1}{\kappa^2}\int d^4x\sqrt{\gamma}\left(3 - \frac{1}{8}e^{2\phi}\partial_i\chi\partial^i\chi\right) - \log v\int d^4x\sqrt{\gamma}\mathcal{A}, \tag{3}$$

where v is the Fefferman-Graham (FG) coordinate, γ is the induced metric on a $v = v_0$ surface, and the limit $v_0 \to 0$ is understood. $\mathcal{A}(\gamma_{ij}, \phi, \chi)$ is the conformal anomaly, which when evaluated on our solution takes the value $\mathcal{A}(\gamma_{ij}, \phi, \chi) = N_c^2 a^4/48\pi^2$.

From the results of [17] the stress tensor is seen to be diagonal, $\langle T_{ij}\rangle = \mathrm{diag}(E, P_\perp, P_\perp, P_\parallel)$, and to obey

$$\partial^i\langle T_{ij}\rangle = 0, \qquad \langle T^i_i\rangle = \mathcal{A}, \tag{4}$$

thus confirming that translation invariance is preserved. $P_\perp$ ($P_\parallel$) are the pressures in the x, y (z) directions. As a consequence of the anomaly the transformation of the stress tensor under a rescaling of a, T contains an inhomogeneous piece [4,18], i.e.,

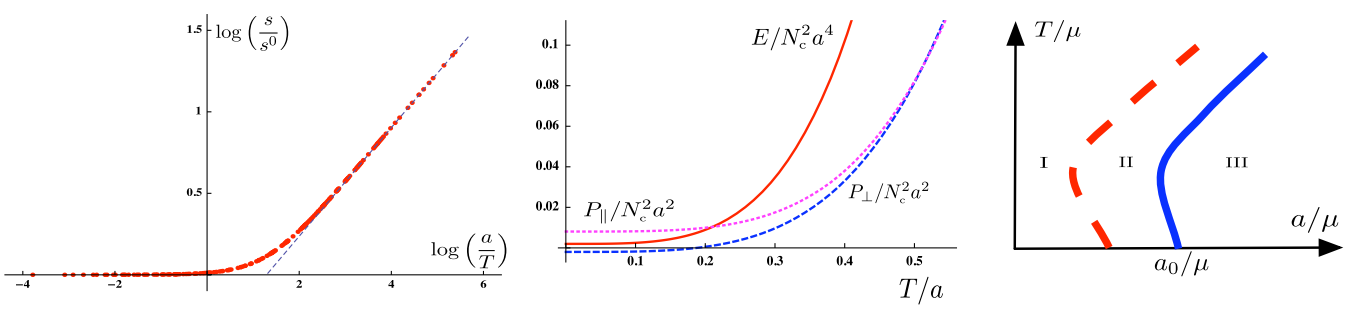

FIG. 1 (color online). (Left) Entropy density as a function of a/T. (Center) Energy and pressures as functions of T/a for fixed $a \simeq 2.86$ and $\log\mu = 1/2$. (Right) Qualitative phase diagram.

$$\langle T_{ij}(ka, kT)\rangle = k^4\langle T_{ij}(a, T)\rangle + k^4 \log k \mathcal{A} h_{ij}, \quad (5)$$

where $h_{ij} = \mathrm{diag}(1, -1, -1, 3)$. In turn, this means that the stress tensor must take the form

$$\langle T_{ij}(a, T)\rangle = a^4 t_{ij}(a/T) + \log(a/\mu)\mathcal{A} h_{ij}, \quad (6)$$

where the arbitrary reference scale μ is a remnant of the renormalization process, much like the subtraction point in quantum chromodynamics (QCD). Different choices of μ are simply different choices of renormalization scheme. We emphasize that the presence of this scale implies that the physics depends on the two dimensionless ratios T/μ and a/μ, not just on T/a. Representative plots of the energy and the pressures are shown in Fig. 1 (center).

Thermodynamics.—As usual, the free energy $F(a, T) = E - Ts = -P_\perp$ is obtained from the on-shell Euclidean action and satisfies $(\partial F/\partial T)_a = -s$ [4,11]. Unlike the entropy density, which is scheme independent, the energy density and the pressures are scheme dependent (i.e., depend on μ), but the thermodynamic relations among them are scheme independent [4]. We recall that the necessary and sufficient conditions for local thermodynamic stability are

$$c_a \equiv T(\partial S/\partial T)_a > 0, \qquad F'' \equiv (\partial^2 F/\partial a^2)_T > 0. \quad (7)$$

Phase diagram.—Approximate analytic solutions can be found in the limits $T \gg a$, μ and $T \ll a$, μ, and these suffice to draw the qualitative phase diagram shown in Fig. 1 (right), which we have also verified numerically [4]. $F''(a, T)$ is negative in zone I and positive in zones II and III. $P_\parallel(a, T) - P^0(T)$ is negative in zones I and II and positive in zone III, with $P^0(T) = \pi^2 N_c^2 T^4/8$ the isotropic pressure. Note that each of the three zones includes points with $T = 0$ as well as points with arbitrarily large a and T.

It follows that the homogeneous phase with uniform $D7$-brane density is in stable thermal equilibrium in zone III. In particular, as can be seen from the positive slope of the continuous red curve in Fig. 1 (center), the specific heat is $c_a > 0$ everywhere. Also, the pressures and the energy are all monotonically increasing functions of T at fixed a, so the speed of sound in all directions is real and positive. There are no thermal instabilities anywhere in the phase diagram.

In contrast, the homogeneous phase is in unstable thermal equilibrium against infinitesimal charge fluctuations in zone I, where the second condition in (7) is violated. In zone II the system is in metastable thermal equilibrium, since it is unstable only against finite charge fluctuations: the pressure in the z direction is smaller than the pressure of the isotropic phase, and thus bubbles of isotropic phase can form and grow, forcing a compression of the charge in the z direction [19]. In other words, in zones I and II a carefully prepared homogeneous system with initial (a, T) will fall apart into a mixed phase consisting of high-density anisotropic "droplets" or "filaments" surrounded by isotropic regions [20]. The local charge density $a' > a$ will be the same in each of the droplets, and the pressure will exactly equal that of the isotropic phase at the same final temperature, $P_\parallel(a', T') = P^0(T')$. The pair (a', T') therefore lies on the continuous blue curve of the phase diagram.

In Landau's theory of phase transitions the homogeneous phase in zones I, II, and III would be described by a saddle point of the free energy with at least one unstable direction, by a metastable local minimum, and by a stable global minimum, respectively.

Discussion.—Our system is in anisotropic thermal equilibrium. This is not surprising, since in the gauge theory isotropy is broken explicitly by an anisotropic external source. The string description makes it clear that the resulting system can be thought of as a fluid with a conserved, isotropy-breaking, two-brane charge (see, e.g., [11]).

It is remarkable that our solution is completely regular despite the fact that it incorporates the full backreaction of the $D7$-branes, whose number scales as $n_{D7} \sim N_c/\lambda$. Relatedly, we note that the parameter controlling their backreaction, $\lambda n_{D7}/N_c$, is coupling enhanced as in the case [21] of flavor $D7$-branes.

The physics in zones I and II shares some similarities with that of QCD at low T and finite baryon density [22]. In that case the pressure of a chirally broken homogeneous phase with density lower than a critical density n_0 is negative (except in a tiny region of very small densities). This indicates an instability towards the formation of "droplets" of higher density n_0 in which $P = 0$ and chiral symmetry is restored, surrounded by empty space with $n = 0$ and $P = 0$. In our case, the role of the chirally restored phase is played by the anisotropic phase, the analogue of n_0 is a_0 [see Fig. 1 (right)], and the "droplets" correspond to the regions of nonzero $D7$-brane density. These similarities suggest that the transition from the mixed phase to the homogeneous phase may occur via a percolation mechanism, as in some QCD models [23] of chiral symmetry restoration [24].

The instabilities we have uncovered are reminiscent of instabilities of anisotropic weakly coupled plasmas [8]. Somewhat pictorially, the main similarity is the tendency to "filamentation," which in weakly coupled plasmas can be understood (very roughly) as the tendency of similarly oriented currents to cluster together. We emphasize though that there are obvious differences. In a weakly coupled plasma the anisotropy is "dynamical" since it arises from the momentum distribution of the particles that compose the plasma. In contrast, in our case the plasma is static and intrinsically anisotropic because of the presence of dissolved extended objects. In any case, we stress that whether a real connection exists between the instabilities studied here and those of weakly coupled plasmas is a question beyond the scope of this Letter.

The instabilities of our solution are also reminiscent of the phenomenon of cavitation, i.e., the formation of bubbles of vapor in regions of a flowing liquid in which the pressure of the liquid drops below its vapor pressure. Cavitation has been proposed [25] (see also [26]) as a mechanism that would lead to fragmentation of the QGP into droplets that would subsequently evaporate, thus providing a new scenario for how hadronization is achieved. In that context the analogue of vapor pressure is the pressure of the vacuum, $P = 0$, whereas in ours it is the pressure of the isotropic phase. As above, however, we emphasize that in the case of [25] the pressure drop is due to a dynamical effect, namely, to the viscosity corrections that result from the expansion of the plasma. In contrast, in our case this is a static effect presumably resulting from the interaction of the extended objects in the plasma.

Note that an instability discovered in [12] does not directly apply here, since Lifshitz (instead of AdS) boundary conditions were assumed in [12].

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