Lasing without Inversion in Circuit Quantum Electrodynamics

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(Received 23 January 2011; published 23 August 2011)

We study the photon generation in a transmission line oscillator coupled to a driven qubit in the presence of a dissipative electromagnetic environment. It has been demonstrated previously that a population inversion in the qubit can lead to a lasing state of the oscillator. Here we show that the circuit can also exhibit the effect of "lasing without inversion." It arises since the coupling to the dissipative environment enhances photon emission as compared to absorption, similar to the recoil effect predicted for atomic systems. While the recoil effect is very weak, and so far elusive, the effect described here should be observable with realistic circuits. We analyze the requirements for system parameters and environment.

DOI: 10.1103/PhysRevLett.107.093901 PACS numbers: 42.55.−f, 03.65.Yz, 85.25.Cp

Introduction.—The basic "circuit quantum electrodynamics" (cQED) system consists of a superconducting qubit coupled to a transmission line oscillator. The former replaces the atom, the latter the radiation field of the traditional quantum electrodynamics setup [1]. In recent experiments [2], a cQED version of the single-atom maser was realized [3–5]. The same setup can be used to create nonclassical photon states in the resonator [6] or, when coupled to a mechanical oscillator, it is one of the prime candidates to observe nonclassical states in macroscopic objects [7,8].

The exchange of energy quanta between an atom and a cavity or, in the present case, a qubit and a resonator, including the escape of photons, is governed by the balance

$$\frac{P_{n+1}}{P_n} = \frac{\Gamma^+_{\rm ph} P_\uparrow + \kappa \bar{n}}{\Gamma^-_{\rm ph} P_\downarrow + \kappa(\bar{n}+1)}. \quad (1)$$

Here P_n is the probability to find n photons in the cavity, while $P_{\uparrow/\downarrow}$ are the occupation probabilities of the atomic levels. The rates of stimulated photon emission and absorption are $\Gamma^+_{\rm ph}$ and $\Gamma^-_{\rm ph}$, respectively, while κ is the decay rate of the oscillator with $\bar{n}$ thermal photons. In thermal equilibrium the occupation probability decreases as a function of the photon number: $P_{n+1}/P_n = \bar{n}/(\bar{n}+1)$. In contrast, the lasing state is characterized by a peak of P_n at nonzero values of n, and hence $P_{n+1} > P_n$ below the peak. In the optical regime the absorption and emission rates are roughly equal. Hence the condition that P_n should grow with n can be met only if a population inversion, $P_\uparrow > P_\downarrow$, is created in a pump process, which typically involves a third level [2,9].

As lasing technology progressed, it became clear that population inversion is not really needed. One of the earliest schemes of lasing without inversion (LWI) is the pumping of Rabi sidebands [10–13]. However, this process is not strictly LWI, since closer inspection reveals that it is based on a population inversion in the dressed states basis [14]. LWI without hidden inversion can be achieved only by breaking the symmetry of photon emission and absorption [15–17].

In this Letter, we will explore a different method that is more similar to the shift of the emission peak caused by the recoil effect on atoms [18]. In this case the atom always absorbs kinetic energy for both optical emission and absorption. The resulting asymmetry for the resonance frequency of emission and absorption makes LWI possible. This effect has never been verified experimentally in conventional lasing setups, since it requires large frequencies and/or low temperatures such that the recoil energy exceeds the temperature [19]. Additionally, the environment should be Markovian and stay in equilibrium. With cQED systems these requirements can be met. One reason is that in cQED there are many intrinsic sources of nonclassical noise suppressing energy emission as compared to absorption. Another is the strong coupling between the qubit and the resonator, which makes it possible to create a lasing situation even in the presence of strong noise.

The system.—The basic mechanism which leads to lasing without inversion is the shift of the photon emission line due to the coupling to a dissipative environment. This effect may play a role in every setup that can be described by the extension of the Jaynes-Cummings Hamiltonian ($\hbar = 1$):

$$H = \omega_0 a^\dagger a + g(a^\dagger \sigma_- + a\sigma_+) + \tfrac{1}{2}\Delta E \sigma_z + \tfrac{1}{2} X(t)\sigma_z. \quad (2)$$

Here a is the annihilation operator for the oscillator with frequency ω_0, the Pauli matrices σ_i act on the states $|\uparrow\rangle$ and $|\downarrow\rangle$, and the qubit energy splitting and the strength of the coupling between oscillator and qubit are denoted by ΔE and g, respectively. We also account for fluctuations $X(t)$ of the energy splitting of the qubit. They may originate from the electrodynamic environment and, together with strong coupling, make inversionless lasing possible. The noise is characterized by the correlator $\langle X(t)X(0)\rangle$, which depends on the effective environmental impedance $Z(\omega)$. We assume it to be dominated by an Ohmic environment coupled capacitively to the system, leading to $Z(\omega)/R_K = \epsilon_C\omega_R/(\omega^2+\omega_R^2)$, parametrized by a coupling strength ϵ_C, cutoff frequency ω_R, and resistance quantum R_K.

Here we consider situations where the noise is strong. Therefore we proceed by using the polaron transformation $U = \exp[-i(\sigma_z/2)\int_{-\infty}^t X(t')dt']$, which yields

$$\begin{aligned} H &= H_0 + H_g,\\ H_0 &= \omega_0 a^\dagger a + \tfrac{1}{2}\Delta E\sigma_z,\\ H_g &= g[\sigma_+ a e^{-i\int_{-\infty}^t X(t')dt'} + \text{H.c.}]. \end{aligned} \tag{3}$$

In the following, we assume that H_g leads to incoherent transitions, which we analyze in the spirit of the so-called $P(E)$ theory [20] by an appropriate Liouville term in the master equation. This treatment of the coupling between the oscillator and qubit differs from the approaches to lasing followed, e.g., by Refs. [4,12,21,22], where the coupling is treated as part of the coherent time evolution. However, as we will demonstrate below, our approach is valid in a broad parameter regime, and it reproduces the results for the average photon number and phase coherence time which are typical for lasing. Additionally, it allows us to describe the shift of the photon emission line due to the environment.

Master equation.—Our system is described by the master equation

$$\dot\rho = i[H_0,\rho] + [\mathcal{L}_{\text{diss}} + \mathcal{L}_{\text{pump}} + \mathcal{L}_g]\rho. \tag{4}$$

It contains three Liouville operators describing dissipation in the oscillator $\mathcal{L}_{\text{diss}}\rho$, an incoherent pump process $\mathcal{L}_{\text{pump}}\rho$, and photon emission and absorption $\mathcal{L}_g\rho$. Dissipation in the oscillator is described by the standard Lindblad operator [22] with decay rate κ. Throughout this Letter, we will assume $\bar n \ll 1$.

Incoherent pumping is described by

$$\begin{aligned} \mathcal{L}_{\text{pump}}\rho = {}&\Gamma_{\text{up}}(\sigma_+\rho\sigma_- - [\sigma_-\sigma_+,\rho]_+/2)\\ &+\Gamma_{\text{down}}(\sigma_-\rho\sigma_+ - [\sigma_+\sigma_-,\rho]_+/2), \end{aligned} \tag{5}$$

where $[,]_+$ denotes an anticommutator. Without coupling to the oscillator, $g=0$, the occupation probabilities of the atom, $P_\uparrow = \langle\uparrow|\rho|\uparrow\rangle$ and $P_\downarrow$, would satisfy $P_\uparrow/P_\downarrow = \Gamma_{\text{up}}/\Gamma_{\text{down}}$. Hence, as long as $\Gamma_{\text{down}} \ge \Gamma_{\text{up}}$, the pumping does not produce a population inversion. For later use we introduce the parameter $\Gamma_{\text{pump}} = \Gamma_{\text{up}} + \Gamma_{\text{down}}$ and define the population inversion coefficient $D_0 = (\Gamma_{\text{up}} - \Gamma_{\text{down}})/\Gamma_{\text{pump}}$. A possible realization of a pump would be the application of classical noise with the spectrum $S(\omega)$, which leads to $\Gamma_{\text{up}} = \Gamma_{\text{down}} \propto S(\Delta E)$.

The crucial effect to be considered here is the emission and absorption of photons, i.e., the transitions between the states $|\uparrow\rangle|n\rangle$ and $|\downarrow\rangle|n+1\rangle$ induced by H_g. For this purpose we expand the time evolution of the density matrix up to second order in g, which yields

$$\begin{aligned} \mathcal{L}_g\rho = {}&\Gamma_{\text{ph}}^+(\sigma_- a^\dagger\rho a\sigma_+ - [a\sigma_+\sigma_- a^\dagger,\rho]_+/2)\\ &+\Gamma_{\text{ph}}^-(2a\sigma_+\rho\sigma_- a^\dagger - [\sigma_- a^\dagger a\sigma_+,\rho]_+/2). \end{aligned} \tag{6}$$

The photon emission and absorption rates $\Gamma_{\text{ph}}^\pm = g^2 S_{\text{ph}}(\pm\delta\omega)$ depend on the spectral function

$$\begin{aligned} S_{\text{ph}}(\omega) &= \int_{-\infty}^{\infty} dt C_{\text{ph}}(t) e^{-\Gamma_{\text{pump}}|t|/2} e^{i\omega t},\\ C_{\text{ph}}(t) &= \exp\bigg\{-\int d\omega \frac{Z(\omega)}{\omega R_K}\\ &\quad\times\bigg[2\sin^2\Big(\frac{\omega t}{2}\Big)\coth\Big(\frac{\omega}{2k_BT}\Big) + i\sin\omega t\bigg]\bigg\}, \end{aligned} \tag{7}$$

at the frequency given by the detuning $\delta\omega = \Delta E - \omega_0$, and temperature T. In the simplest form the spectral function and correlator $C_{\text{ph}}(t)$ are well known from the $P(E)$ theory. Here, we also account for a level broadening effect due to the pumping process. The extension can be derived in the frame of the diagrammatic expansion of the time evolution of the density matrix [23].

We assume that the qubit is pumped fairly strongly, as was the case in the experiments of Ref. [2]. In this case the lowest-order expansion in H_g converges for all combinations of ϵ_C and ω_R, as long as $g\sqrt{n} \ll \Gamma_{\text{pump}}$. Our approach reproduces the standard lasing results in the limit of weak noise $Z(\omega)\to 0$. Furthermore, even for $g\sqrt{n} > \Gamma_{\text{pump}}$ our master equation remains valid for $\omega_R \to \infty$ in the stationary limit and for $g^2 n \ll \sqrt{k_BT\epsilon_C}\omega_R$ at all times.

The master equation (4) can be solved by numerical diagonalization in the eigenbasis of H_0. In addition, there are standard methods to solve it analytically [21]. One is based on the equation of motion of operator averages. Another relies on an adiabatic elimination of qubit states from which an effective equation of motion for $\rho_{nn'} = \langle n|\text{Tr}_{\text{qubit}}\rho|n'\rangle$ in the basis of oscillator photon number states is derived. The trace is taken over the qubit eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$. To find a closed set of equations for the operator averages, we make use of the semiclassical approximation $\langle a^\dagger a\sigma_z\rangle \approx \langle a^\dagger a\rangle\langle\sigma_z\rangle$. This is a good approximation in the stationary limit and allows us to derive an analytical expression for the average photon number $\langle n\rangle = \langle a^\dagger a\rangle$. To describe the time evolution of the system, we use the adiabatic elimination of the qubit degrees of freedom. This method accurately describes the time evolution of the off-diagonal matrix elements even for $\Gamma_{\text{ph}}^+ \neq \Gamma_{\text{ph}}^-$.

Lasing without inversion.—The effects of inversionless lasing in cQED are demonstrated in Fig. 1, which shows the average photon number in the oscillator for three cases. The cases displayed correspond to the standard lasing situation in the limit of vanishing noise, $Z(\omega) \to 0$, and to situations with strong noise. In the standard case the number of photons starts to rise only as inversion is established, $D_0 > 0$, in contrast to the cases with strong noise where large photon numbers are excited already without inversion.

To achieve inversionless lasing, we tune the system such that $\delta\omega = \Delta E - \omega > 0$. If a photon is created, the additional energy is given to the environment. For the reverse process, the environment has to provide the energy $\delta\omega$. At low temperatures, this process is strongly suppressed, and hence there is a strong imbalance between the emission and the absorption rates: $\Gamma^+_{\rm ph} \gg \Gamma^-_{\rm ph}$ [see Fig. 1(a)]. This imbalance is the origin of the inversionless lasing. The optimal condition for inversionless lasing is reached if the coupling to noise is stronger than the level broadening caused by the pump: $\epsilon_C \gg \Gamma_{\rm pump}$. Furthermore, the asymmetry between the rates is maximized for $\epsilon_C \gg k_B T \gtrsim \omega_R$. In this limit the maximum strength of photon emission is obtained for $\delta\omega \approx \epsilon_C$ [20].

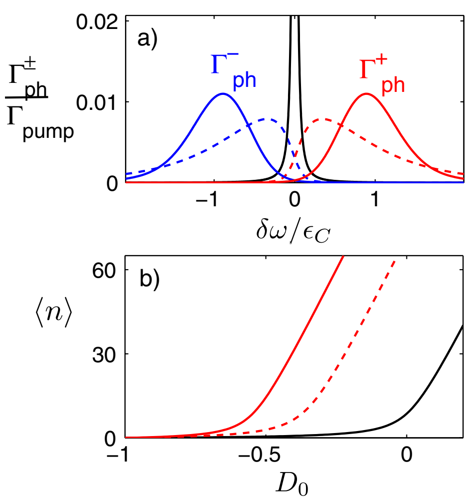

FIG. 1 (color online). (a) Photon emission and absorption rates $\Gamma^+_{\rm ph}$ and $\Gamma^-_{\rm ph}$, respectively, as a function of the detuning $\delta\omega$. Without noise, which can be effectively modeled by $\omega_R \to \infty$, the two rates are equal (black line). Dashed red and blue lines show the emission and absorption rate, respectively, for $\omega_R/\epsilon_C = 0.5$. Full red and blue lines show the rates for $\omega_R/\epsilon_C = 0.05$. (b) The average photon number $\langle n \rangle$ as a function of the inversion coefficient D_0. The black line is the result for a qubit without noise, $\omega_R \to \infty$, at $\delta\omega = 0$. The red lines are the results under the influence of noise, for $\omega_R = 0.5$ at $\delta\omega/\epsilon_C \approx 0.35$ (dashed line) and for $\omega_R = 0.05$ at $\delta\omega/\epsilon_C \approx 1$ (full line). The frequency detuning has been chosen to maximize the photon number. We used the parameters $\Gamma_{\rm pump}/\epsilon_C = 0.0325$, $k_B T/\epsilon_C = 0.05$, $g/\epsilon_C = 0.01$, and $\kappa/\epsilon_C = 8.125 \times 10^{-5}$.

In thermal equilibrium the rates obey detailed balance, and the photon absorption rate $\Gamma^-_{\rm ph}$ is exponentially suppressed. However, the pumping creates an extra broadening [see Eq. (7)]. Hence, for the optimum conditions we can approximate the rates by $\Gamma^+_{\rm ph} \approx g^2/\sqrt{k_B T \epsilon_C}$ and $\Gamma^-_{\rm ph} \approx g^2 \Gamma_{\rm pump}/\epsilon_C^2$. From these rates we can formulate the threshold condition defined in Eq. (1) for $\bar{n} \ll 1$ as

$$\frac{P_{n+1}}{P_n} = \frac{1}{\sqrt{k_B T}\Gamma_{\rm pump}} \frac{g^2 \epsilon_C^{3/2} \Gamma_{\rm up}}{g^2 \Gamma_{\rm down} + \kappa \epsilon_C^2}. \quad (8)$$

This relation allows us to quantify the two requirements for optimized inversionless lasing, i.e., strong noise at low temperatures and strong coupling. Comparing the absorption and emission rates, we find for the former

$$\epsilon_C^{3/2}/k_B T \gg \Gamma_{\rm pump}. \quad (9)$$

The second condition requires a comparison between photon emission and oscillator decay rates and yields

$$g^2/\sqrt{\epsilon_C k_B T} \gg \kappa. \quad (10)$$

For the derivation of the conditions (9) and (10), we assumed $\sqrt{k_B T \epsilon_C} \gg \omega_R$. However, this is not strictly necessary as is shown in Fig. 2, where we show the average photon number obtained from a numerical evaluation of the photon emission and absorption rates. Even for large cutoff frequencies we can create a lasing state in the

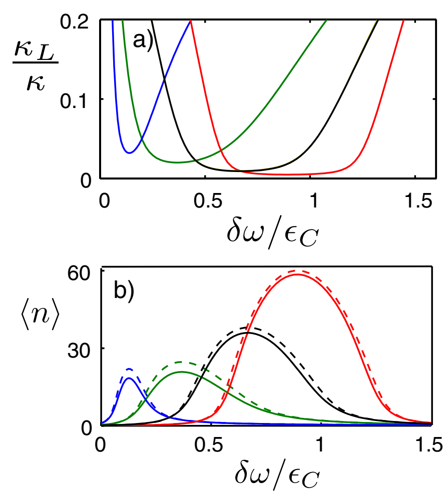

FIG. 2 (color online). (a) The phase correlation decay rate κ_L as a function of the detuning $\delta\omega$. The values of the cutoff frequency are $\omega_R/\epsilon_C = 0.05$ (red line), $\omega_R/\epsilon_C = 0.2$ (black line), $\omega_R/\epsilon_C = 0.5$ (green line), and $\omega_R/\epsilon_C = 2$ (blue line). (b) The average photon number $\langle n \rangle$ as a function of the frequency detuning $\delta\omega$ with color coding as in (a). Full lines display the results of a numerical solution of the master equation; dashed lines are the approximate analytical solution (11). The system parameters are $\Gamma_{\rm pump}/\epsilon_C = 0.0325$, $k_B T/\epsilon_C = 0.05$, $g/\epsilon_C = 0.01$, $\kappa/\epsilon_C = 8.125 \times 10^{-5}$, and $D_0 = -0.25$.

oscillator. However, the photon number reaches a maximum for a low cutoff frequency.

For unequal photon creation and annihilation rates, and provided that all conditions formulated in Eqs. (8)–(10) apply, we find the average photon number to be

$$\langle n\rangle = \frac{\Gamma_{up}\Gamma_{ph}^{+} - \Gamma_{down}\Gamma_{ph}^{-}}{(\Gamma_{ph}^{+} + \Gamma_{ph}^{-})\kappa} + \mathcal{O}(\kappa^0). \tag{11}$$

This results confirms our previous discussion. To get a high photon number we need a low decay rate κ in the oscillator and $\Gamma_{up}\Gamma_{ph}^{+} - \Gamma_{down}\Gamma_{ph}^{-} > 0$. For equal photon creation and annihilation rates, $\Gamma_{ph}^{+} = \Gamma_{ph}^{-}$, we would need $\Gamma_{up} > \Gamma_{down}$ to get a large photon number. If all orders of κ are taken into account, our result for the average photon number exactly reproduces the standard lasing results in the limit $Z(\omega) \to 0$.

One of the characteristic properties of a laser is the phase coherence, as expressed by the slow decay of the phase correlator $C(t) = \langle a(t)a(0)\rangle$. In the Markovian limit the correlator decays exponentially: $C(t) \propto e^{-\kappa_L t}$, with rate κ_L. After adiabatic elimination of the time evolution of the qubit, the problem is reduced to the decay of the matrix elements ρ_{nn+1} [21]. For a large photon number the decay rate is given by

$$\kappa_L = \frac{1}{\langle n\rangle}\frac{(\kappa\Gamma_{pump} + \Gamma_{down}\Gamma_{ph}^{-} + \Gamma_{up}\Gamma_{ph}^{+})}{8\Gamma_{pump}}. \tag{12}$$

Even for $\Gamma_{ph}^{+} \neq \Gamma_{ph}^{-}$ the decay rate is inversely proportional to the photon number, as is usual for a laser. For weak noise $Z(\omega) \to 0$, this result reproduces the standard lasing results, as long as we stay within the limit of validity of our master equation. For a discussion of the phase correlation decay rate in the limit of strong coupling and weak noise, see, e.g., Ref. [5].

In Fig. 2, we show the phase correlation decay rate κ_L and the average photon number $\langle n\rangle$ as a function of the frequency detuning $\delta\omega$. The numerical and analytical results for the average photon number are in good agreement; for the decay rate the comparison is not shown but there is also qualitative agreement. The results shown for different values of the cutoff frequency ω_R demonstrate that the enhancement of the photon number is stronger the smaller ω_R is. However, as ω_R becomes smaller we also reach the limit of the validity of our master equation. On the other hand, even for larger values of ω_R, lasing is possible with a strongly reduced phase correlation decay rate.

Discussion.—The lasing mechanism discussed in this Letter can qualitatively be described as result of a population inversion in the environment. In this picture, inversion is established between the excited state with no population in the environment and the ground state with a quickly decaying population in some environmental modes. However, since for a Markovian bath with a smooth spectral function the population of the environmental modes cannot be defined in a consistent way, this analysis does not allow a quantitative discussion, in contrast to the analysis presented above based on the shift of the emission and absorption peaks [14,18,19].

A specific system where the conditions for lasing without inversion can be realized is the single artificial-atom laser investigated by Astafiev *et al.* [2], consisting of a charge qubit coupled to an oscillator. An applied transport voltage induces quasiparticle tunneling via a third state, which can be used to create a population inversion in the qubit, which has led to the observed lasing. With the same setup it would also be possible to only create an enhanced population of the excited qubit level without inversion. The noise spectrum needed for inversionless lasing can be realized by coupling the charge qubit to an external resistor (see, e.g., Ref. [24]). To reach a low cutoff frequency it is necessary to use a resistor with a large resistance, as it has been demonstrated, e.g., in Ref. [25]. The strong coupling needed to satisfy the requirement (10) is standard in most cQED experiments [1,2,26].

We acknowledge stimulating discussions with V. Brosco and Y. Nakamura. Y. U. acknowledges support by the Strategic International Cooperative Program of the Japan Science and Technology Agency (JST) and the German Science Foundation (DFG).

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