Gravity and Large Black Holes in Randall-Sundrum II Braneworlds

Pau Figueras[1] and Toby Wiseman[2]

[1]DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom
[2]Theoretical Physics Group, Blackett Laboratory, Imperial College, London SW7 2AZ, United Kingdom

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We show how to construct low energy solutions to the Randall-Sundrum II (RSII) model by using an associated five-dimensional anti–de Sitter space (AdS_5) and/or four-dimensional conformal field theory (CFT_4) problem. The RSII solution is given as a perturbation of the AdS_5-CFT_4 solution, with the perturbation parameter being the radius of curvature of the brane metric compared to the AdS length ℓ. The brane metric is then a specific perturbation of the AdS_5-CFT_4 boundary metric. For low curvatures the RSII solution reproduces 4D general relativity on the brane. Recently, AdS_5-CFT_4 solutions with a 4D Schwarzschild boundary metric were numerically constructed. We modify the boundary conditions to numerically construct large RSII static black holes with radius up to $\sim 20\ell$. For a large radius, the RSII solutions are indeed close to the associated AdS_5-CFT_4 solution.

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Introduction.—The single brane Randall-Sundrum II (RSII) model [1,2] is remarkable in that it is claimed to yield 4D low energy physics for brane observers even though the 5D geometry is not compact. By using arguments from anti–de Sitter/conformal field theory (AdS/CFT), it has been claimed that the low energy behavior of this model for a brane observer is equivalent to 4D gravity coupled to a conformal field theory [3–7]. A remarkable conjecture was then made in Refs. [8–10] that static black holes cannot exist in RSII for a radius much greater than the AdS length ℓ, although we note this is based on free field theory intuition, which may not hold [11,12]. By using the numerical methods of Refs. [13,14], black holes in 5D RSII with a radius up to $\sim 0.2\ell$, and for 6D up to $\sim 2.0\ell$, were constructed in Refs. [15–17]. However, by using the same methods, it has subsequently been argued that even very small RSII static black holes do not exist [18,19].

In this Letter, we first will make precise the claim that low energy physics on the brane is described by gravity coupled to a CFT. We shall explicitly show how to construct low curvature solutions to RSII, including matter on the brane, from an associated AdS_5-CFT_4 problem, where the boundary metric is given by a particular perturbation of the brane metric. An AdS_5-CFT_4 solution with a Schwarzschild boundary metric has recently been numerically constructed by us and Lucietti [20], and in the second half of the Letter we shall report on work where we modify the numerical construction used there to compute the RSII black hole solutions for both large and small radii.

Low curvature RSII solutions from AdS_5-CFT_4.—In this section we will follow Ref. [6], although we note that the emphasis is subtly different. Our aim is not to derive an effective 4D description of gravity on the brane as done in Ref. [6] but rather to explicitly demonstrate the relation between solutions in AdS/CFT and corresponding ones in RSII.

Consider a solution to AdS_5-CFT_4 with boundary metric $g^{(0)}_{\mu\nu}$. The 5D metric g_{AB} obeying $R_{AB} = -\frac{4}{\ell^2} g_{AB}$ can be written as

$$ds^2 = g_{AB}dx^A dx^B = \frac{\ell^2}{z^2}[dz^2 + \tilde{g}_{\mu\nu}(z,x)dx^\mu dx^\nu] \quad (1)$$

near the conformal boundary $z = 0$, where the Fefferman-Graham expansion dictates that

$$\tilde{g}_{\mu\nu}(z,x) = g^{(0)}_{\mu\nu}(x) + z^2[R^{(0)}_{\mu\nu}(x) - \tfrac{1}{4}g^{(0)}_{\mu\nu}(x)R^{(0)}(x)] + z^4[g^{(4)}_{\mu\nu}(x) + t_{\mu\nu}(x)] + 2z^4 \log z\, h^{(4)}_{\mu\nu}(x) + O(z^6), \quad (2)$$

where the expressions for $g^{(4)}$ and $h^{(4)}$ can be found in Ref. [21]. Here $g^{(0)}_{\mu\nu}(x)$ and $t_{\mu\nu}(x)$ are the two constants of integration for the bulk equations which are second order in z. The constraint equations for this radial evolution imply $\nabla^{(0)}_\mu t^{\mu\nu} = 0$ and $t = \frac{1}{16}[R^{(0)}_{\alpha\beta}R^{(0)\alpha\beta} - \frac{1}{3}(R^{(0)})^2]$, and $t_{\mu\nu}$ gives the vacuum expectation value of the CFT_4 stress tensor as $\langle T^{\rm CFT}_{\mu\nu}\rangle = t_{\mu\nu}/(4\pi\ell G_5)$.

We assume that for some boundary metric $g^{(0)}_{\mu\nu} = g_{\mu\nu}$ a solution exists for boundary conditions in the IR of the geometry such that the metric tends to the Poincaré horizon of AdS. We further assume that solutions exist for regular perturbations of the boundary metric $g^{(0)}_{\mu\nu} = g_{\mu\nu} + \epsilon^2 h_{\mu\nu}$ in some finite neighborhood of $\epsilon = 0$, so that $t_{\mu\nu}[g + \epsilon^2 h] = t_{\mu\nu}[g] + O(\epsilon^2)$.

From this AdS_5-CFT_4 solution we will construct an RSII solution in the limit where brane curvatures are small compared to the curvature of the bulk AdS_5. We take two copies of the solution above restricted to $z \geq \epsilon$ and glue them together on their common boundary. We then identify the two halves under a $\mathbb{Z}_2$ action which leaves the orbifold plane $z = \epsilon$, the RSII brane with induced metric $\gamma_{\mu\nu}$, fixed. The Israel conditions determine the matter on the brane to have stress tensor

$$8\pi G_4 T^{\rm brane}_{\mu\nu} = \frac{2}{\ell}\left(K_{\mu\nu} - K\gamma_{\mu\nu} + \frac{3}{\ell}\gamma_{\mu\nu}\right), \quad (3)$$

where $K_{\mu\nu} = -\frac{1}{2}\frac{z}{\ell}\partial_z[\frac{\ell^2}{z^2}\tilde{g}_{\mu\nu}(z,x)]$ is the extrinsic curvature of the $z=\epsilon$ surface.

We begin the construction by choosing the perturbation $h_{\mu\nu}$ so that $\gamma_{\mu\nu} = \frac{\ell^2}{\epsilon^2} g_{\mu\nu}$, and then

$$g^{(0)}_{\mu\nu} = g_{\mu\nu} + \frac{\epsilon^2}{2}\left(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\right) + O(\epsilon^4 \log\epsilon). \quad (4)$$

It is convenient to work with the rescaled brane metric $g_{\mu\nu}$ rather than $\gamma_{\mu\nu}$, since we are interested in the limit $\epsilon \to 0$.

Computing the brane matter stress tensor from (3) in terms of the rescaled brane metric $g_{\mu\nu}$ gives the "Einstein equation on the brane" derived in Ref. [6]:

$$\begin{aligned} G_{\mu\nu} - 8\pi G_4 T^{\rm brane}_{\mu\nu} &= \epsilon^2 \log\epsilon\, b_{\mu\nu}[g] + \epsilon^2(16\pi G_4\langle T^{\rm CFT}_{\mu\nu}[g]\rangle \\ &\quad + a_{\mu\nu}[g]) + O(\epsilon^4\log\epsilon), \end{aligned} \quad (5)$$

where (the separately conserved) tensors $a_{\mu\nu}$ and $b_{\mu\nu}$ are, respectively,

$$\begin{aligned} a_{\mu\nu}[g] &\equiv -\tfrac{1}{4}\nabla^2 R_{\mu\nu} + \tfrac{1}{12}\nabla_\mu\nabla_\nu R + \tfrac{1}{24}\nabla^2 \bar{R} g_{\mu\nu} + \tfrac{1}{6}RR_{\mu\nu} \\ &\quad + \tfrac{1}{8}R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} - \tfrac{1}{24}R^2 g_{\mu\nu} - \tfrac{1}{2}R_{\mu\alpha\nu\beta}R^{\alpha\beta}, \\ b_{\mu\nu}[g] &\equiv -\tfrac{1}{2}\nabla^2 R_{\mu\nu} + \tfrac{1}{6}\nabla_\mu\nabla_\nu R + \tfrac{1}{12}\nabla^2 R g_{\mu\nu} + \tfrac{1}{3}RR_{\mu\nu} \\ &\quad + \tfrac{1}{4}R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} - \tfrac{1}{12}R^2 g_{\mu\nu} - R_{\mu\alpha\nu\beta}R^{\alpha\beta}. \end{aligned} \quad (6)$$

The parameter ϵ controls the curvature scale on the brane relative to the AdS length, and $\epsilon \to 0$ gives the low curvature limit on the brane where we see the usual 4D Einstein equations are recovered.

Subject to the assumption that the AdS_5-CFT_4 solution exists for boundary metric (4), we have constructed a braneworld solution with metric $\gamma_{\mu\nu} = \frac{\ell^2}{\epsilon^2}g_{\mu\nu}$ perturbatively in ϵ. Working to higher order in ϵ one will obtain further local higher curvature terms together with terms involving functional derivatives of the CFT_4 stress tensor. We note that we have not assumed $g_{\mu\nu}$ is a metric perturbation of flat space, only that its curvature is everywhere small.

Interestingly, the leading correction in ϵ for 4D Einstein gravity comes from the $O(\epsilon^2\log\epsilon)$ local four-derivative term $b_{\mu\nu}[g]$. In the absence of brane matter, $T^{\rm brane}_{\mu\nu} = 0$; then as $g_{\mu\nu}$ is Ricci flat to order $O(\epsilon^0)$, the corrections $a_{\mu\nu}$ and $b_{\mu\nu}$ vanish to give

$$\delta G_{\mu\nu} = 16\pi G_4\langle T^{\rm CFT}_{\mu\nu}[g]\rangle, \quad (7)$$

where $G_{\mu\nu}[g] = \epsilon^2\delta G_{\mu\nu} + O(\epsilon^4)$. This form of correction was conjectured by Ref. [9], and here we have provided a proof of this, although we emphasize that, by including brane matter, the CFT correction is not the leading one.

5D static RSII black holes.—Setup.—In the previous section we have seen how low curvature classical solutions of the RSII model with brane metric $\frac{\ell^2}{\epsilon^2}g_{\mu\nu}$ are related to the existence of AdS_5-CFT_4 solutions with a boundary metric a perturbation of $g_{\mu\nu}$. Consider large static vacuum black holes in RSII. Provided there exists a static AdS_5-CFT_4 solution with a 4D Schwarzschild as the boundary metric and which asymptotes to the Poincaré horizon of AdS in the IR, then large black holes in the RSII scenario exist. Furthermore, these will be static, since the AdS_5-CFT_4 solution they derive from has boundary metric (4) with g being a Schwarzschild so that $g^{(0)}$ is static (and will be to all orders in ϵ) and the bulk geometry must inherit the isometries of the boundary metric [22].

Such an AdS_5-CFT_4 solution has recently been found [20] by using the new numerical approach of Ref. [23]. In the remainder of this Letter, we will report on work where we modify this numerical construction to replace the AdS boundary ("UV" end of the geometry) with an RSII brane boundary condition and solve the resulting elliptic boundary value problem. The details will be presented in a longer forthcoming paper.

Following Ref. [23] we analytically continue our static solution to the Euclidean signature and consider the solution to the 5D Einstein-DeTurck equations with a negative cosmological constant:

$$R_{MN} + \frac{4}{\ell^2}g_{MN} - \nabla_{(M}\xi_{N)} = 0 \quad (8)$$

where $\xi^M = g^{PQ}(\Gamma^M_{PQ}\text{-}\bar{\Gamma}^M_{PQ})$, Γ^M_{PQ} is the connection associated to the metric g_{AB} that we want to determine, and $\bar{\Gamma}^M_{PQ}$ is a connection associated to a fixed reference metric $\bar{g}$. For the Euclidean signature, the above equation is elliptic and can be solved as a boundary value problem for well-posed boundary conditions.

An important point is that a solution to this Einstein-DeTurck equation need not be Einstein if $\xi^A \neq 0$. In favorable situations, one can analytically show that solutions with nonzero ξ^A, called "Ricci solitons," cannot exist [20]. However, even if they may exist, provided the elliptic problem and boundary conditions are well-posed, solutions should be locally unique. Hence, an Einstein solution cannot be arbitrarily close to a soliton solution [24], and one should easily be able to distinguish the Einstein solutions of interest from solitons.

Following Ref. [20], we will choose a similar ansatz to that used for the AdS_5-CFT_4 solution with a Schwarzschild boundary, namely,

$$\begin{aligned} ds_5^2 &= \frac{\ell^2}{\Delta(r,x)^2}\left(r^2 T d\tau^2 + \frac{x^2 g(x) S}{f(r)^2}d\Omega^2_{(2)} + \frac{4A}{f(r)^4}dr^2 \right. \\ &\quad \left. + \frac{4B}{f(r)^2 g(x)}dx^2 + \frac{2rxF}{f(r)^3}drdx\right), \\ \Delta(r,x) &= \frac{(1-r^2) + \tilde{\beta}(1-x^2)}{\tilde{\beta}(1-r^2)}, \end{aligned} \quad (9)$$

where $f(r) = 1 - r^2$ and $g(x) = 2 - x^2$, and $X = \{T, S, A, B, F\}$ are smooth functions (to be determined) which depend on (r, x) only. The (dimensionless) coordinates (r, x) both take values in the range $[0, 1]$, and we assume $T, S > 0$ and that $AB - r^2x^2g(x)F^2/16 > 0$ to ensure that the metric is Euclidean with the correct topology. In contrast to the setting in Ref. [20], now the function $\Delta(r, x)$ does *not* vanish at $x = 1$, so there is no UV conformal boundary there. We choose the reference metric $\bar{g}$ to be (9) with $T = A = B = S = 1$ and $F = 0$.

The boundaries of our domain are the same as in Ref. [20] (and therefore so are the boundary conditions for the functions X [25]), except that now $x = 1$ corresponds to the location of the brane. Here we impose the vacuum Israel matching conditions [Eq. (3) with the left-hand side equal to zero] together with $\xi_x = 0$ and $F = 0$, which imply mixed Neumann-Dirichlet conditions for the various functions X. Such boundary conditions have been considered in Ref. [20], where they were shown to give a regular elliptic system. Furthermore, they imply $\partial_n \xi_r = \frac{2}{\ell}\xi_r$ on the brane (where ∂_n denotes the normal derivative), which is compatible with obtaining an Einstein solution with $\xi = 0$ everywhere. Note that imposing the Israel vacuum condition and both $\xi_r = 0$ and $\xi_x = 0$ on the brane does not give a regular elliptic system [20,24]. We remark that for this negative tension orbifold brane there is no maximum principle argument that rules out the existence of a soliton solution. Hence we will have to check explicitly that our solution is Einstein and not a soliton—indeed, we have found no solitons.

Finally, we note that our metric (9) has the dimensionless parameter $\tilde{\beta}$ which determines the inverse temperature as $\beta = 4\pi\tilde{\beta}\ell$. This effectively controls the size of the black hole relative to the cosmological constant scale.

Results.—Two approaches have been proposed in Ref. [23] to solve (8). The Ricci flow method works particularly well in finding the AdS_5-CFT_4 solution in Ref. [20] since the solution is a stable fixed point of the flow. All the RSII black holes we have found have a *single* Euclidean negative mode, and hence the solution is an unstable fixed point of the Ricci flow which makes this method less practical. For this reason we have used the Newton algorithm to find solutions. We have used two independent codes: One is based on a pseudospectral collocation approximation in r, x (up to 40×40 points), and the other is based on second-order finite difference. As expected, the former gives highly accurate results, and the data presented are for this code. The finite difference code gives consistent, but less accurate, solutions for the resolutions attainable.

To construct black holes whose proper radius on the brane, R_4, is large compared to ℓ [for instance, setting $\tilde{\beta} = 20$ in (9)], we found that using the reference metric $\bar{g}$ as the initial guess was sufficient for Newton's method to converge. Once a large black hole had been obtained, we could easily find nearby ones by simply perturbing both the previous solution and the reference metric varying $\tilde{\beta}$. Using this procedure we have been able to construct braneworld black holes with $R_4/\ell \in [0.07, 20]$. It should be possible to extend this range by increasing the resolution, but we have not attempted to do so.

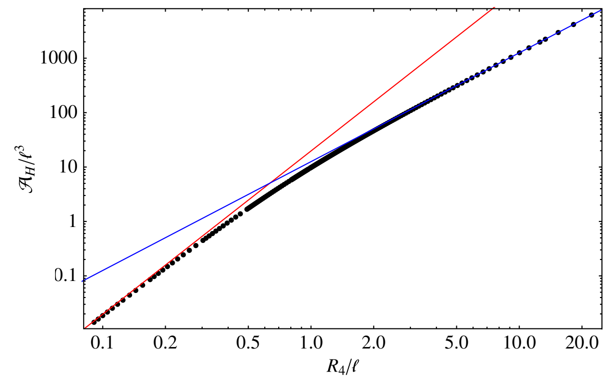

FIG. 1 (color online). Area of the black hole as a function of the radius of the horizon on the brane (black dots) and the same quantity for an asymptotically flat Schwarzschild black hole in 5D (red line) and in 4D (blue line). Note the log scale of both axes.

In Fig. 1, we have plotted the area of the full 5D black hole as a function of the radius of the horizon on the brane, comparing it with the analogous quantities for an asymptotically flat Schwarzschild black hole in 5D (red) and 4D (blue), respectively. It is apparent from this plot that small (compared to ℓ) braneworld black holes behave like 5D asymptotically flat Schwarzschild black holes and large ones recover 4D behavior.

We have embedded the geometry of the spatial cross sections of the horizon into $\mathbb{H}^4$, $ds^2 = \frac{\ell^2}{z^2}(dz^2 + dr^2 + r^2 d\Omega^2_{(2)})$, as a surface of revolution $r(z)$ such that the induced metric on this surface is that of the horizon. To compare black holes of different sizes we have fixed the maximum extent of the horizon into the bulk to be at $z = 1$ so that the brane is located at a $z = z_{\min}$ which depends on the size of the black hole. Figure 2 depicts the embeddings of the horizon of

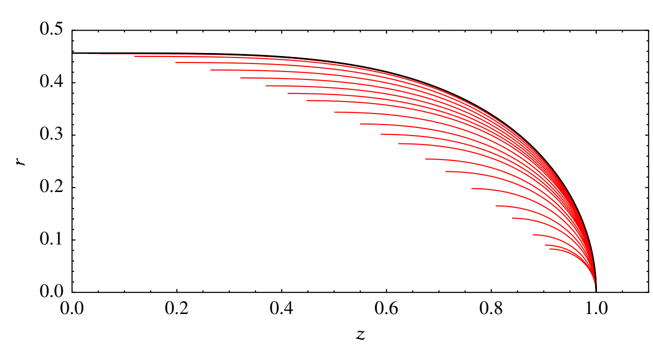

FIG. 2 (color online). Embedding of the spatial cross sections of the horizon into $\mathbb{H}^4$ (red). The black curve corresponds to the embedding of the horizon of the AdS_5-CFT_4 solution of Ref. [20], with a 4D Schwarzschild as the conformal boundary metric.

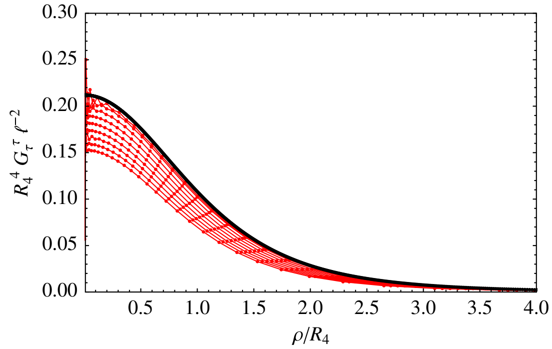

FIG. 3 (color online). $R_4{}^4 G_\tau{}^\tau \ell^{-2}$ computed from the induced geometry on the brane against the proper distance for $R_4/\ell \sim$ 1.24–6.70 (red). In black, the right-hand side of (7) computed from the solution of Ref. [20]. The red curves approach the black one as the black hole size is increased. For large black holes, the actual value of $G_\tau{}^\tau$ on the brane is so small as to be comparable to the numerical error, and we see some noise in this quantity.

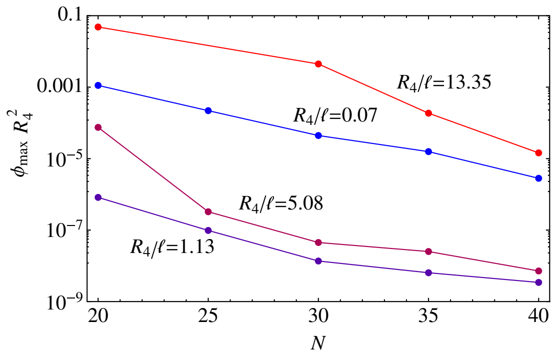

FIG. 4 (color online). Maximum value of ϕ over domain for $R_4/\ell = 13.35, 5.04, 1.13, 0.07$, as a function of the number of grid points N for the pseudospectral code.

braneworld black holes of different sizes (red), together with the embedding of the AdS$_5$-CFT$_4$ solution of Ref. [20]. This gives a beautiful graphical confirmation of the analysis given in the first part of this Letter. For large black holes, where $z_{\min} \rightarrow 0$, we see the horizon tends to that of the AdS$_5$-CFT$_4$ solution, the perturbation from it getting smaller as the cutoff $z_{\min}$ is removed.

We can provide evidence of dynamical stability for our solution by computing the spectrum of our linearized Euclidean Einstein-DeTurck equation about our solutions. We note that this linear operator must be computed anyway as part of the Newton method. For transverse traceless perturbations about an Einstein solution, it coincides with the spectrum of the Lichnerowicz operator restricted to static axisymmetric modes [23]. We find that for all our solutions there is a single negative mode, which for small R_4/ℓ tends to the usual negative mode of 5D asymptotically flat Schwarzschild black hole. Small solutions are close to 5D Schwarzschild and should be stable. The absence of any zero modes and hence new negative modes as one moves to larger radius solutions indicates we should expect no (axisymmetric) dynamical instabilities.

As the black hole becomes larger, the induced geometry on the brane tends to the 4D asymptotically flat Schwarzschild solution. We may verify this by computing the induced Einstein tensor on the brane. In Fig. 3, we plot the dimensionless quantity $R_4{}^4 G_\tau{}^\tau \ell^{-2}$ against the proper radial distance from the horizon along the brane, ρ, in the combination ρ/R_4. The other components of $G_\mu{}^\nu$ give the same behavior, and we see that the solutions become Ricci flat with corrections going as $O(\ell^2/R_4^2)$, i.e., $O(\epsilon^2)$. With these scalings we see the curves for large radius solutions limit to a fixed curve, which appears to be precisely predicted by the stress tensor of the AdS$_5$-CFT$_4$ solution of Ref. [20]. This explicitly confirms the prediction (7).

Finally, we comment on the possibility that our solutions are in fact Ricci solitons. We have performed convergence tests which indicate that the solutions indeed have $\phi = \xi^A \xi_A \rightarrow 0$ in the continuum limit. As shown in Fig. 4 for black holes with $R_4/\ell = O(1)$, our 40×40 pseudospectral code gives a maximum value of ϕ, denoted by $\phi_{\max}$, such that $\phi_{\max} < 10^{-8}$, which is already very small. It is also worth noting that, for a fixed spatial resolution, $\phi_{\max}$ grows as the black hole becomes very large or very small, which is expected since we have to resolve widely separated length scales, namely, the horizon radius on the brane, and the AdS length.

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