Dissonance is Required for Assisted Optimal State Discrimination

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The roles of quantum correlations, entanglement, discord, and dissonance needed for performing unambiguous quantum state discrimination assisted by an auxiliary system are studied. In general, this procedure for conclusive recognition between two nonorthogonal states relies on the availability of entanglement and discord. However, we find that there exist special cases for which the procedure can be successfully achieved without entanglement. In particular, we show that the optimal case for discriminating between two nonorthogonal states prepared with equal *a priori* probabilities does not require entanglement but quantum dissonance only.

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It was believed that entanglement was the only kind of quantum correlation contained in a bipartite quantum system needed to carry out quantum information tasks. However, it has been found that in some cases, deterministic quantum computation can be done in the absence of entanglement [1]. Specifically, the correlations of separable mixed states are a key ingredient for the speedup [2]. Additionally, it has been realized that protocols such as quantum nonlocality and quantum search can be realizable even with separable states [3,4]. Thus, entanglement does not account for all possible quantum correlations contained in a bipartite state. In spite of this, entanglement is the only quantum correlation for pure states; recently it has been recognized that this is not generally true for mixed states. Ollivier and Zurek [5] introduced the quantum discord which captures the nonclassical correlations, including but not limited to entanglement. Thus entanglement and discord are fundamental resources that allow us to perform some quantum information processes with or without classical counterparts.

Discord is attracting interest for studying separable states exhibiting quantum correlations other than entanglement [6–8], and their evolutions under decoherent mechanisms [9–11]. In general, discord has to be obtained numerically; however, for certain classes of mixed states it is even possible to calculate it analytically [12,13]. Discord can be interpreted as a distance measure between the studied mixed state ρ and its closest classical one χ_ρ. Similarly, entanglement is the distance from ρ to the closest separable state σ_ρ [14]. In a unified view of quantum and classical correlations, quantum dissonance as a distance measure between σ_ρ and its closest classical state χ_{σ_ρ} was introduced. In other words, the dissonance is a quantum correlation with excluded entanglement [14]. The role of quantum correlations in carrying out quantum information protocols is an issue deserving fundamental interest in light of these recent findings.

On the other hand, unambiguous discrimination among linearly independent nonorthogonal quantum states is a problem of fundamental interest [15–19]. Two nonorthogonal states require a three-dimensional Hilbert space for implementing an optimal procedure of unambiguous state discrimination [20]. When the states are codified strictly in a two-dimensional Hilbert space, like a spin-$1/2$ particle, the process for unambiguous discrimination has to be assisted by an ancillary system in order to increase the dimension of the Hilbert space [20]. Naively thinking, entanglement would be the main ingredient for performing the assisted state discrimination protocol [16]. However, considering the above discussion, the question is, what kind of correlations, entanglement, quantum discord, or dissonance, are behind a successful discrimination outcome?

Consider that a qubit is randomly prepared in one of the two nonorthogonal states $|\psi_+\rangle$ or $|\psi_-\rangle$ with *a priori* probabilities p_+ and $p_- = 1 - p_+$, respectively [16,21]. Let us assume that the system can be coupled to an auxiliary qubit A by a joint unitary transformation U such that

$$U|\psi_+\rangle|k\rangle_a = \sqrt{1-|\alpha_+|^2}|+\rangle|0\rangle_a + \alpha_+|0\rangle|1\rangle_a, \qquad (1a)$$

$$U|\psi_-\rangle|k\rangle_a = \sqrt{1-|\alpha_-|^2}|-\rangle|0\rangle_a + \alpha_-|0\rangle|1\rangle_a, \qquad (1b)$$

where $|k\rangle_a$ is a known initial state and $\{|0\rangle_a, |1\rangle_a\}$ is an orthonormal basis of the auxiliary system. We have also considered the orthonormal basis $\{|0\rangle, |1\rangle\}$ of the principal system and the orthonormal states $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ [16].

The *a priori* fixed overlap $\langle\psi_+|\psi_-\rangle = \alpha = |\alpha|e^{i\theta}$ does not change due to the joint unitary transformation; thus, from (1) we see that α_+ and α_- probability amplitudes satisfy the constraint $\alpha = \alpha_+^*\alpha_-$, i.e., $|\alpha_-| = |\alpha|/|\alpha_+|$ and $\theta_- - \theta_+ = \theta$, with $\theta_\pm$ the phases of $\alpha_\pm$, and $|\alpha| \leq |\alpha_+| \leq 1$. Thus, the α_+ amplitude defines the joint unitary transformation which allows us to couple the quantum system of interest with the auxiliary one. We must consider that in principle the relative phase θ could be managed by properly choosing the axes on the Bloch sphere. As we will see, one convenient choice would be $\theta = 0$. In this manner, after applying the unitary U we have the mixed states

$$\rho_{|\alpha_+|} = p_- U|\psi_-\rangle|k\rangle\langle\psi_-|\langle k|U^\dagger + p_+ U|\psi_+\rangle|k\rangle\langle\psi_+|\langle k|U^\dagger. \tag{2}$$

This expression reveals in principle the presence of quantum correlations between the system and the ancilla.

The process of discriminating unambiguously the prepared initial states $|\psi_+\rangle$ or $|\psi_-\rangle$ is achieved by performing a von Neumann measurement on the basis $\{|0\rangle_a, |1\rangle_a\}$ of the ancillary system. The recognition is successful when the ancilla is projected onto the state $|0\rangle_a$, since in this case the system of interest collapses to the orthogonal states $|+\rangle$ or $|-\rangle$, depending on in which state, $|\psi_+\rangle$ or $|\psi_-\rangle$, it was initially prepared. Otherwise, the process fails when the projection is onto $|1\rangle_a$. In this case, the initial information disappears since the principal system collapses into $|0\rangle$, whatever the prepared state is.

The probability of success depends on the $|\alpha_+|$ parameter and is given by

$$P_s(|\alpha_+|) = 1 - p_- \frac{|\alpha|^2}{|\alpha_+|^2} - p_+|\alpha_+|^2. \tag{3}$$

Notice that $P_s(|\alpha_+|)$ is different from zero for any value of $|\alpha_+|$. This means that this process always allows discriminating probabilistically and unambiguously the prepared state. The optimal success probability is attained for $|\alpha_+| = \sqrt[4]{p_-/p_+}\sqrt{|\alpha|}$, or $|\alpha_+| = |\alpha|$ $(p_+ \geq p_-)$, or $|\alpha_+| = 1$ $(p_- \geq p_+)$, and can be expressed as

$$P_{s,\max} = \begin{cases} 1 - 2\sqrt{p_+ p_-}|\alpha| & \text{if } 0 \leq |\alpha| \leq \bar{\alpha} \\ (1 - |\alpha|^2)\max\{p_+, p_-\} & \text{if } \alpha \leq |\bar{\alpha}| \leq 1, \end{cases} \tag{4}$$

where $\bar{\alpha} = \min\{\sqrt{p_+/p_-}, \sqrt{p_-/p_+}\}$. It is worth emphasizing that for $|\alpha| \in [0, |\bar{\alpha}|[$ both states could be recognized and the probability is linear in $|\alpha|$. For $|\alpha| \in [|\bar{\alpha}|, 1[$, only one state can be discriminated, say, $|\psi_+\rangle$ $(|\psi_-\rangle)$ if $p_+ \geq p_-$ $(p_+ \leq p_-)$ and the probability is quadratic in $|\alpha|$. In addition, we note that, as is known, the probability is 1 for discriminating two orthogonal states $(\alpha = 0)$, whereas it is 0 when the two states are different only by a phase factor $(|\alpha| = 1)$.

We now answer our main question about what kind of correlation allows performing the procedure of conclusive nonorthogonal state discrimination when it is assisted by an auxiliary system. We can ask first how much entanglement between the systems is required. The amount of entanglement contained in state (2) is given by the concurrence [22]

$$C(\rho_{|\alpha_+|}) = \Bigg\{2\Bigg(\sqrt{1-|\alpha_+|^2}|\alpha_+|p_+ + \sqrt{1-\frac{|\alpha|^2}{|\alpha_+|^2}}\frac{|\alpha|}{|\alpha_+|}p_-\Bigg)^2 - 8|\alpha| \times\sqrt{1-|\alpha_+|^2}\sqrt{1-\frac{|\alpha|^2}{|\alpha_+|^2}}p_+p_-\cos^2\frac{\theta}{2}\Bigg\}^{1/2}. \tag{5}$$

We see that the concurrence depends on the phase θ of the overlap α, and it has its minimum value when θ is zero. The maximal concurrence holds for $\theta = \pm\pi$, which corresponds just to the average concurrence of the decomposition (2). This is illustrated in Fig. 1, where concurrence is shown as a function of $|\alpha_+|$ for different values of θ, $|\alpha|$, and p_+. The solid line is the probability $P_s(|\alpha_+|)$ of Eq. (3). It is clear from this picture that we cannot relate the probability of success to a given amount of entanglement, given that, for different values of θ attaining different values of entanglement, we have the same probability of success. Even more, for $\theta = 0$ there are some values of $|\alpha_+|$ for which the entanglement is zero. In particular we note that there is one zero of entanglement around the maximal probability of discrimination for $p_+ = p_-$. For $p_+ \neq p_-$, still we have successful discrimination with zero entanglement but with nonoptimal probability. These facts indicate that the discrimination processes do not necessarily require entanglement.

We now consider the optimal process of state discrimination (the maximum value of P_s), for which the concurrence is obtained by evaluating (5) in $|\alpha_+| = \sqrt[4]{p_-/p_+}\sqrt{|\alpha|}$, when $|\alpha| \in [0, \bar{\alpha}]$, or in $|\alpha_+| = |\alpha|$ $(|\alpha_+| = 1)$ when $|\alpha| \in [\bar{\alpha}, 1]$ and $p_+ \geq p_-$ $(p_+ \leq p_-)$. The concurrence $C(\rho_{\sqrt[4]{p_-/p_+}\sqrt{|\alpha|}})$ is symmetric under the

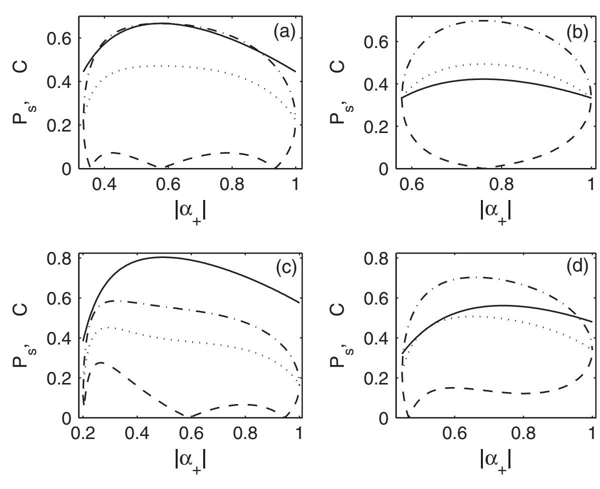

FIG. 1. Concurrence $C(\rho_{|\alpha_+|})$ as a function of $|\alpha_+|$ for different values of θ: 0 (dashed lines), $\pi/2$ (dotted lines), π (dash-dotted lines). The solid line is the probability $P_s(|\alpha_+|)$. The *a priori* probability $p_+ = 1/2$ for (a) and (b) and $p_+ = 2/5$ for (c) and (d). The corresponding overlaps $|\alpha|$ are (a) $1/3$, (b) $1/\sqrt{3}$, (c) $1/5$, and (d) $1/\sqrt{5}$.

exchange of p_+ and p_- and it takes the maximal value for $\theta = \pi$ and the minimal one for $\theta = 0$ ($|\alpha| \in [0, \bar{\alpha}[$), whereas $C(\rho_{|\alpha|})$ and $C(\rho_1)$ do not depend on the phase θ ($|\alpha| \in [\bar{\alpha}, 1]$). These features are illustrated in Figs. 2(a) and 2(b) which show the concurrence as functions of $|\alpha|$ for different values of θ and p_+, i.e., $C(\rho_{\sqrt[4]{p_-/p_+}\sqrt{|\alpha|}})$ in the interval $|\alpha| \in [0, \bar{\alpha}]$ and $C(\rho_1)$ in the interval $|\alpha| \in [\bar{\alpha}, 1]$. In the optimal success probability the concurrence can be zero only when $p_+ = p_-$ and $\theta = 0$, as is illustrated by the solid line in Fig. 2(b). In this case the concurrence is given by

$$C(\rho_{\sqrt{|\alpha|}}) = \sqrt{2|\alpha|(1 - |\alpha|)}\left|\sin\frac{\theta}{2}\right|. \quad (6)$$

It is clear from this expression that for $\theta = 0$ there is an optimal discrimination process without assistance of entanglement with $P_s = 1 - |\alpha|$.

In the general case $C(\rho_{|\alpha_+|})$, expression (5) can exhibit other zeros only when $\theta = 0$. Specifically, we get that concurrence (5) is zero when $|\alpha_+|$ is a root of the 4th degree polynomial in $|\alpha_+|^2$, given by

$$|\alpha_+|^8 - |\alpha_+|^6 + |\alpha|^2 \frac{p_-^2}{p_+^2}|\alpha_+|^2 - |\alpha|^4 \frac{p_-^2}{p_+^2} = 0. \quad (7)$$

A simple analytical solution of this equation is found for equal *a priori* probabilities, namely,

$$|\alpha_+|_{C=0} = \sqrt{|\alpha|}, \qquad 0 \le |\alpha| \le 1, \quad (8a)$$

$$|\alpha_+|_{C=0} = \sqrt{\frac{1 \pm \sqrt{1 - 4|\alpha|^2}}{2}}, \qquad 0 \le |\alpha| \le \frac{1}{2}. \quad (8b)$$

Equation (8a) coincides with the case of optimal success probability. In the solution (8b) the process of discrimination happens with constant probability $P_s = 1/2$, as can be seen by replacing (8b) in (3). Figure 3 shows the solutions $|\alpha_+|$ of Eq. (7) as a function of $|\alpha|$ for which the concurrence is equal to zero. The solutions given in (8) are shown in Fig. 3(a), where the solid line corresponds to the (8a) solution, dashed line to the (8b) solution with plus sign, and dotted line to (8b) solution with minus sign. The case $p_+ \neq p_-$ is illustrated in Figs. 3(b)–3(d) where also three solutions appear in one interval and one solution in another interval.

From the previous analysis we learned that the assisted state discrimination process can be performed in the absence of entanglement. In those cases it is important to know which correlation is behind the state recognition. In this respect, recent progress in the understanding of correlations other than entanglement, such as quantum discord, dissonance, or the classical one, can shed light to answer this question. In the absence of entanglement in a mixed state, quantum dissonance is present if discord is different from zero [14]. If discord is zero, then only classical correlations could be present [14]. As is well known, quantum discord for a bipartite mixed ρ_{AB} is given by [5]

$$D = I(\rho_{AB}) - \sup_{\{\hat{M}_x\}}\left\{S(\rho_A) - \sum_x p_x S(\rho_x)\right\}, \quad (9)$$

where $I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ is the quantum mutual information and $S(\rho)$ is the von Neumann entropy. The second term on the right-hand side of this expression corresponds to the classical correlations. The supreme is taken over all the measurement sets $\{\hat{M}_x\}$ applied on system B, p_x is the probability for outcomes $\hat{M}_x$, and ρ_x is the partial projection of ρ_{AB} defined as $\rho_x = \mathrm{Tr}_B[(\mathbf{1}_A \otimes \hat{M}_x)\rho_{AB}]/p_x$ [5]. In this way, discord can be calculated numerically. However, there are some cases where the optimization problem was solved analytically [12,13].

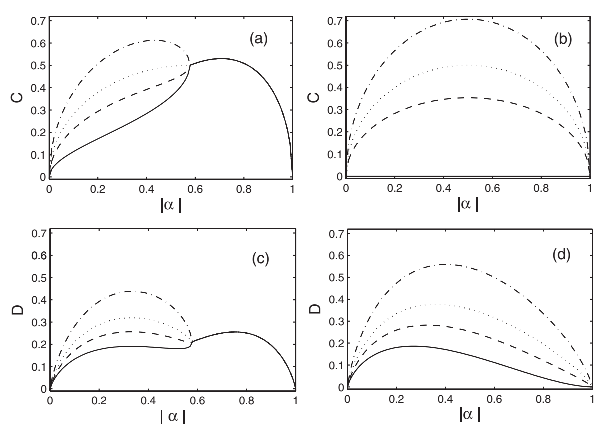

FIG. 2. Concurrence (a),(b) and discord (c),(d) as functions of $|\alpha|$ for the case of optimal success probability of discriminating. We consider different values of θ: 0 (solid lines), $\pi/3$ (dashed lines), $\pi/2$ (dotted lines), and π (dot-dashed lines) in the interval $|\alpha| \in [0, \bar{\alpha}]$ and solid line in the interval $|\alpha| \in [\bar{\alpha}, 1]$. In (a),(c) $p_+ = 1/4$ and in (b),(d) $p_+ = 1/2$.

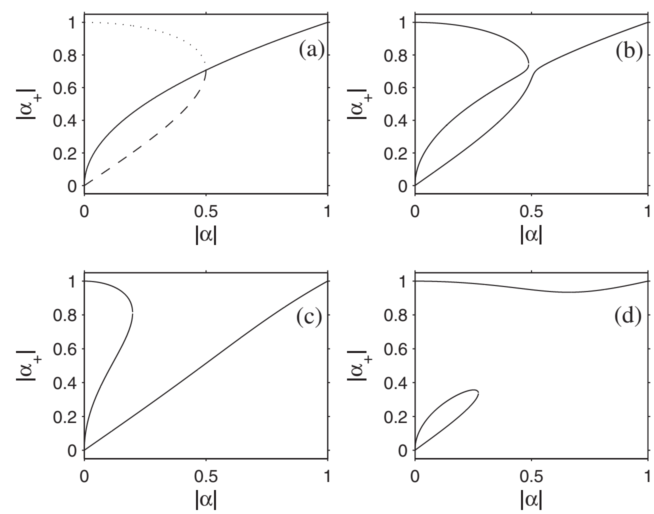

FIG. 3. $|\alpha_+|$ solutions as functions of $|\alpha|$ for which the concurrence is zero. In (a) $p_+ = 1/2$, (b) $p_+ = 499/1000$, (c) $p_+ = 1/3$, (d) $p_+ = 3/5$.

In our study, we avoid the optimization problem by using the Koashi-Winter identity [23] since the rank-two state (2) can be written as a tripartite pure state:

$$|\Psi\rangle = \sqrt{p_+}(U|\alpha_+\rangle|k\rangle_a)|0\rangle_c + \sqrt{p_-}(U|\alpha_-\rangle|k\rangle_a)|1\rangle_c, \quad (10)$$

where we introduced a fictitious qubit C that, once traced, led to the mixed state (2). Following the Koashi-Winter [23] method, we have that $J_{SA} = \min_{\{\hat{M}_x\}}\{\sum_x p_x S(\rho_x)\} = E(\rho_{SC})$, where $E(\rho_{SC})$ is the entanglement of formation between the system of interest and the fictitious qubit C. Then, for calculating the optimization it is easier to calculate $E(\rho_{SC}) = -x\log_2 x - (1-x)\log_2(1-x)$ with $x = (1+\sqrt{1-C_{SC}^2})/2$ and C_{SC} being the concurrence of the reduced ρ_{SC} density matrix. Thus, the quantum discord is given by

$$D(\rho_{SA}) = S(\rho_A) - S(\rho_{SA}) + E(\rho_{SC}). \quad (11)$$

This expression can be calculated more easily than (9). In Fig. 2(c), the discord is shown for the optimal probability of success when $p_+ = 1/4$, and can be compared qualitatively with the corresponding concurrence in Fig. 2(a). We realize that discord also depends on the phase θ when $\alpha \in [0, \bar{\alpha}[$, whereas for $\alpha \in [\bar{\alpha}, 1]$ it does not depend on θ. For the case $p_+ = 1/2$, a similar dependence on θ is shown for discord in Fig. 2(d) as compared with concurrence in Fig. 2(b). In general, we cannot say which correlation is responsible for the state discrimination process. However, we can say that in the optimal case with equal *a priori* probabilities and $\theta = 0$, the process is assisted exclusively by dissonance. Similarly, for the roots (8b), the nonoptimal case, the quantum dissonance can be calculated by using Eq. (11). Figure 4 shows the quantum dissonance as a function of $|\alpha_+|$ for $p_+ = p_-$ and for solutions in (8) with $\theta = 0$. Notice that in all of these three cases the quantum dissonance is responsible for successfully completing the procedure.

One can show that there are always solutions of Eq. (7), some of them illustrated in Fig. 3, for which the process of state discrimination is assisted only by dissonance and not by entanglement.

In summary, we have shown that the protocol for unambiguous discrimination of two nonorthogonal quantum states, assisted by an auxiliary system, in general requires quantum correlations in order to be implemented. The particular case with optimal probability of success requires both entanglement and discord except the case with equal *a priori* probabilities, which is performed with zero entanglement and nonzero discord; i.e., only quantum dissonance is needed in this important case. We also found other nonoptimal state discrimination procedures with different *a priori* probabilities which are assisted by quantum dissonance only, since entanglement is absent and the success probability is different from zero. In other words, here we have found that an assisted unambiguous state discrimination protocol always can be implemented successfully aided only by quantum dissonance. Finally, we would like to emphasize that the optimal assisted state discrimination protocol with equal *a priori* probabilities does not make use of an entangled state but of a nonclassical separable state in general. However, there are two cases for which a classical state appears: (1) two orthogonal states for which the probability of discrimination is one and (2) two parallel states for which the probability of discrimination is zero.

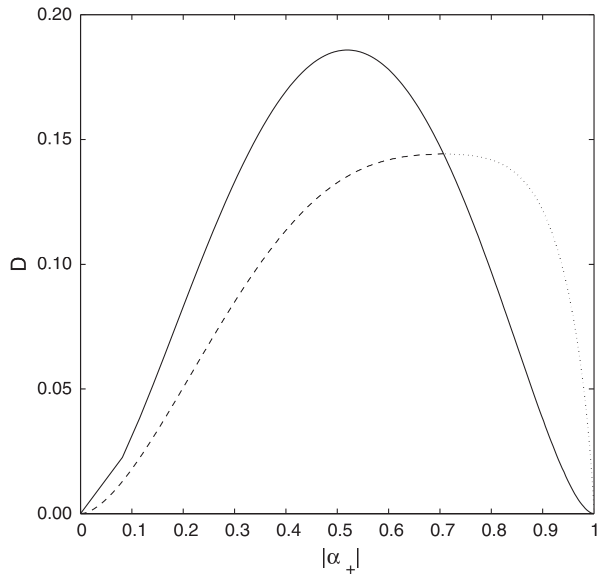

FIG. 4. Quantum dissonance as a function of $|\alpha_+|$ for $p_+ = p_-$ and $\theta = 0$, for solutions in (8). Solid line corresponds to (8a), dashed line to (8b) with minus sign, and dotted line to (8b) with plus sign.

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