Torsional Response and Dissipationless Viscosity in Topological Insulators

Taylor L. Hughes, Robert G. Leigh, and Eduardo Fradkin
Department of Physics, University of Illinois, 1110 West Green St, Urbana, Illinois 61801, USA
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We consider the viscoelastic response of the electronic degrees of freedom in 2D and 3D topological insulators (TI's). Our primary focus is on the 2D Chern insulator which exhibits a bulk dissipationless viscosity analogous to the quantum Hall viscosity predicted in integer and fractional quantum Hall states. We show that the dissipationless viscosity is the response of a TI to torsional deformations of the underlying lattice geometry. The viscoelastic response also indicates that crystal dislocations in Chern insulators will carry momentum density. We briefly discuss generalizations to 3D which imply that time-reversal invariant TI's will exhibit a quantum Hall viscosity on their surfaces.

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A striking feature of a topological insulator (TI) is its topological response. The paradigmatic example is the time-reversal breaking integer quantum Hall effect (IQHE) in 2D which exhibits a Hall conductance that is an integer multiple of e^2/h [1]. It was shown that there exist related states in 3D which are time-reversal invariant [2] and exhibit a topological magnetoelectric effect [3] (TME). While the electromagnetic (EM) response of topological insulators is the most well known, in this Letter we consider the viscoelastic response of the electronic degrees of freedom in TI's. We want to consider the stress response

$$\langle T^{ij}\rangle = \Lambda^{ijk\ell}u_{k\ell} + \eta^{ijk\ell}\dot{u}_{k\ell} \tag{1}$$

where T^{ij} is the stress tensor, Λ, η are the elasticity and viscosity tensors, respectively, and u_{ij} is the strain tensor. Here we show there is a dissipationless viscosity response in the topological Chern insulator [4] (CI) state analogous to that found in the IQHE and fractional QHE states [5–10]. While viscosities are normally associated with frictional dissipation, this viscosity, present only when time-reversal symmetry is broken, implies a force perpendicular to the fluid motion similar to the Lorentz force.

In a condensed matter system the electronic stress response can be calculated by coupling the electronic Hamiltonian to perturbations of the background lattice geometry. The topological responses due to geometric curvature have been studied in Refs. [11,12] in the language of quantum field theory anomalies. Alternatively, we consider the response of topological insulators to an external torsion field. A heuristic understanding of the difference between curvature and torsion is that when an object traverses a small loop in real space it is rotated if there is nonzero curvature, and translated if there is nonzero torsion. A familiar manifestation of torsion is a crystal dislocation. These line defects are singular sources of torsion, analogous to a localized magnetic flux line. For example, while dragging an electron around a magnetic flux line its wave function is multiplied by a $U(1)$ phase, while for a dislocation line it is multiplied by a translation operator along the Burgers vector. We derive the CI viscoelastic response as a linear response to torsional perturbations of the underlying material geometry and mention the response of 3D time-reversal invariant topological insulators (3DTI's).

To understand the torsion response we will often draw comparisons to the well-known EM responses of topological insulators which we briefly review now. The responses of the CI and 3DTI to external EM fields are encapsulated in topological effective actions, i.e., free-energy functionals derived from calculating a partition function in the presence of external fields. The QHE and TME are encoded in the effective actions

$$S_{\text{eff}}^{(\text{QHE})}[A_\mu] = \frac{ne^2}{2h}\int d^3x\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \tag{2}$$

$$S_{\text{eff}}^{(\text{TME})}[A_\mu] = \frac{e^2}{4h}\int d^4x\theta\epsilon^{\mu\nu\sigma\tau}\partial_\mu A_\nu\partial_\sigma A_\tau, \tag{3}$$

respectively, which are derived from the responses of the topological insulators to an external field A_μ, and in 3D an inhomogeneous scalar "axion" field θ (note that n is an integer). The nominal current response is $\langle j^\mu\rangle = \delta S_{\text{eff}}/\delta A_\mu$ which gives the QHE and TME when acting on Eqs. (2) and (3), respectively. All known topological EM responses in various dimensions are described by similar topological effective actions [3].

Our primary interest is the 2D CI for which we will use a continuum massive Dirac Hamiltonian as a model. We couple the massive Dirac Hamiltonian to geometric perturbations, but because of its spinor nature the Dirac Hamiltonian does not couple to geometry through the metric tensor, but instead via the orthonormal triad $\mathbf{e}^a$ and its inverse $\mathbf{e}_a$ (frame field) and the spin connection $\omega^a{}_b$. The Latin index a labels the particular vector of the frame which, when expanded in terms of a local coordinate basis $\partial/\partial x^\mu = (\partial_t, \partial_x, \partial_y)$, has components e_a^μ. In a lattice version of the theory, the frame is defined by the local orbital orientation. The stress response can be thought of as a functional of e^a and $\omega^a{}_b$, but we should not take them to be related to each other as they would be in Riemannian geometry [13]. In the context of condensed matter physics, it is convenient, in fact, to set the spin connection to zero such that the torsion is contained in the properties of the triad alone.

The action and Hamiltonian for continuum 2D massive Dirac fermions coupled to a frame field are

$$\begin{aligned} S &= \int d^3x \det(\mathbf{e})\psi^\dagger\gamma^0(p_\mu e^\mu_a\gamma^a - m)\psi, \\ H &= p_x e^x_a\Gamma^a + p_y e^y_a\Gamma^a + m\Gamma^0, \end{aligned} \tag{4}$$

with $a = 0, 1, 2$, $\gamma^a = (\sigma^z, i\sigma^y, -i\sigma^x)$ and $\Gamma^a = (\sigma^z, \sigma^x, \sigma^y)$. If the frame field is position independent the energy spectrum is simply $E_\pm = \pm\sqrt{p_1^2 + p_2^2 + m^2}$ with $p_a = e^i_a p_i$. This is a gapped insulator when $m \neq 0$. Now we will calculate the off-diagonal response of the stress-energy current (analogous to σ_{xy}) due to a perturbation of the triad $e^a_\mu(x) = \delta^a_\mu + \delta e^a_\mu(x)$ around the trivial background. We will see later that the triad has a simple interpretation in terms of elasticity theory and provides a natural geometric deformation. The stress-energy current that couples to the triad is $T^\mu_a = \frac{1}{\det(\mathbf{e})}\frac{\delta S}{\delta e^a_\mu} = \bar\psi p^\mu\gamma_a\psi$. We wish to integrate out the massive fermions to get an effective action which is a functional of the triad. We are only interested in the terms which lead to dissipationless transport and we find, at leading order,

$$\langle T^\mu_a T^\nu_b\rangle(q) = \frac{1}{16\pi}\eta_{ab}\epsilon^{\mu\nu\sigma}q_\sigma I_T(m), \tag{5}$$

$$I_T(m) = \int_0^\infty dy y \frac{\partial}{\partial y}\frac{m}{(y + m^2)^{1/2}}, \tag{6}$$

where $\eta_{ab} = \mathrm{diag}[1, -1, -1]$, q is the external momentum, and $y = \vec{p}^2$ where p is an internal loop momentum. If we Fourier transform this kernel leads to

$$S_{\mathrm{eff}}[e^a_\mu] = \frac{1}{32\pi}I_T(m)\int d^3x\epsilon^{\mu\nu\rho}e^a_\mu\partial_\nu e^b_\rho\eta_{ab} \tag{7}$$

which is similar to Eq. (2), i.e., a Chern-Simons (CS) term for the triad. Restoring the spin connection, the integral in Eq. (7) is the Lorentz invariant integral $\int e^a \wedge T^b\eta_{ab}$, with T^a the torsion 2-form [13]. For reasons we will see below, we call this a quantum Hall viscosity response.

When probed by an electric field the 2D continuum Dirac model is notorious for having a half-integer QHE ($\sigma_{xy} = \mathrm{sgn}(m)e^2/2h$) which is connected to the parity anomaly [14]. However, when properly regularized, (e.g., on a lattice) σ_{xy} becomes quantized in integer units, as it must for a noninteracting system [4]. In the present case, in the continuum limit, the coefficient $I_T(m)$ is divergent. If we simply cut off the momentum integral at a UV scale Λ then we find $I_T(m, \Lambda) = -m\Lambda + 2m^2\mathrm{sgn}m + O(1/\Lambda)$. Comparing to the quantized Hall conductance, this is quite different, although from symmetry and dimensional analysis there is no choice: this term must break time reversal and thus is an odd function of m. Additionally since e^a_μ is dimensionless (unlike A_μ) this coefficient must have units of $1/[\mathrm{length}]^2$. Hence the leading term is proportional to m and the only other scale Λ. The other unusual thing is that this term is continuous at $m = 0$, unlike the Hall conductance, which jumps. To get physically sensible answers for the Hall viscosity, and the Hall conductance, which cannot be a half-integer, we must more carefully regulate the theory. Here, we describe the standard Pauli-Villars technique with a set of N massive regulator fields, which is appropriate since it preserves all the symmetries of the Hamiltonian. The ith regulator field has mass M_i and wave function renormalization C_i, and we take $M_0 = m$, $C_0 = 1$. The regulated Hall conductance and viscosity are

$$\sigma_{xy} = \frac{e^2}{2h}\sum_{i=0}^N C_i\mathrm{sgn}M_i, \qquad \zeta_{\mathrm{reg}} = \frac{1}{16\pi}\sum_{i=0}^N C_i I_T(M_i), \tag{8}$$

respectively. We can rewrite

$$I_T(M) = -\frac{M}{\sqrt{\pi}}\int_\epsilon^\infty dt t^{1/2}\int_0^\infty dy y e^{-t(y+M^2)} \tag{9}$$

which yields $I_T(M) = -2M/\sqrt{\pi\epsilon} + 2|M|^2\mathrm{sgn}M + O(\sqrt{\epsilon})$. We have three physical constraints that will give proper renormalization conditions: (i) $\sigma_{xy} = ne^2/h$, $n \in \mathbb{Z}$, (ii) ζ_{reg} is finite, and (iii) if $\sigma_{xy} = 0$ then $\zeta_{\mathrm{reg}} = 0$. We know that when the Dirac mass switches sign we go through a phase transition between a trivial insulator and a topological insulator. The sign of m that gives the topological insulator is regularization dependent, and without loss of generality we pick $m > 0$ to be the nontrivial insulator. So for $m < 0$ we require both $\sigma_{xy} = \zeta_{\mathrm{reg}} = 0$, which is the origin of (iii). A solution for M_i, C_i under the constraints above is always possible. In all cases, one finds that in the nontrivial phase we have

$$\sigma_{xy} = \frac{e^2}{h}, \qquad \zeta_{\mathrm{reg}} = \frac{\hbar}{8\pi}\left(\frac{|m|}{\hbar v_F}\right)^2 \equiv \frac{\hbar}{8\pi\ell^2}. \tag{10}$$

If we had chosen $m < 0$ to be the topological phase then the signs of both coefficients would be flipped. Note that we have restored the units in the viscosity response. The dimensions of this coefficient are angular momentum density, which is equivalent to momentum per unit length, and the units of dynamic viscosity (force/velocity). Comparing to the value of the Hall viscosity for the IQHE [5,6] $\zeta_{\mathrm{IQHE}} = \frac{\hbar}{8\pi\ell_B^2}$ we see a similar structure coming from the length scale endowed by the time-reversal breaking field. Also, the viscosity is continuous in the limit $m \to 0$ analogous to the $B \to 0(\ell_B \to \infty)$ limit of the IQHE.

For an easier comparison to the existing literature on the quantum Hall viscosity we will rederive the Hall viscosity for the CI state using an adiabatic transport calculation [5–7]. This can be carried out by putting the Dirac equation on a torus and calculating the adiabatic curvature due to shear deformations (equivalently deformations of the modular parameter τ) of the torus [5]. Consider a square torus, made in $\mathbb{R}^2$ by identifications $(x, y) \sim (x + a, y + b)$ with $a, b \in \mathbb{Z}$, with fixed unit volume, and consider area preserving diffeomorphisms, corresponding to spatial metrics of the form

$$g_{ij} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} \frac{|\tau|^2}{\tau_2} & -\frac{\tau_1}{\tau_2} \\ -\frac{\tau_1}{\tau_2} & \frac{1}{\tau_2} \end{pmatrix}. \tag{11}$$

The basis vectors and the spatial part of the triad are

$$e_1 = \sqrt{\tau_2}\partial_x, \qquad e_2 = \frac{1}{\sqrt{\tau_2}}(-\tau_1\partial_x + \partial_y) \tag{12}$$

$$e^1 = \frac{1}{\sqrt{\tau_2}}(dx - \tau_1 dy), \qquad e^2 = \sqrt{\tau_2}dy \tag{13}$$

respectively, and the Hamiltonian is

$$H = \begin{pmatrix} m & \mathcal{P} \\ \bar{\mathcal{P}} & -m \end{pmatrix} \tag{14}$$

where $\mathcal{P} = \frac{1}{\sqrt{\tau_2}}(\bar{\tau}p_1 - p_2)$. We define $\mathcal{P}\bar{\mathcal{P}} \equiv ||\mathcal{P}||^2$.

We consider the ground state in which all of the negative energy states $\psi_-(p_1, p_2; \tau)$ are occupied. The adiabatic connection can be calculated from the explicit form of the single-particle wave functions and we find

$$A = i\sum_{m,n\in\mathbb{Z}}\psi_-^\dagger(m, n; \tau)d\psi_-(m, n; \tau)$$
$$= -i\sum_{m,n\in\mathbb{Z}} f(||\mathcal{P}||^2)\frac{1}{2}d\ln\left(\frac{\mathcal{P}}{\bar{\mathcal{P}}}\right) \tag{15}$$

where

$$f(||\mathcal{P}||^2) = \frac{m}{(m^2 + ||\mathcal{P}||^2)^{1/2}} \tag{16}$$

and where the sums are over the discrete quantized momenta on the torus. This gives the adiabatic curvature

$$F = i\frac{d\tau\wedge d\bar{\tau}}{2\tau_2}\sum_{m,n}p_1^2 f'(||\mathcal{P}||^2). \tag{17}$$

If we convert the sum into an integral we find

$$F = i\frac{d\tau\wedge d\bar{\tau}}{\tau_2^2}\frac{I_T(m)}{16\pi} \tag{18}$$

which yields the same (unregulated) viscosity as above.

We will now give a physical interpretation in terms of conventional elasticity fields [15]. Assuming we have an elastic medium, we can pick a reference undisplaced state and define a (spacetime) displacement field $u^a(x)$. Then the triad can be written as $e^a_\mu = \delta^a_\mu + w^a_\mu$ where $w^a_\mu = \partial u^a/\partial x_\mu$ is the distortion tensor [15]. To simplify we assume that $u^0 \equiv 0$, i.e., $e^a = dt$. Now w^a_μ contains the velocity field $w^A_0 = v^A$ and the spatial distortion tensor w^A_i where $A = 1, 2$ and $i = x, y$. This formulation of the triad in terms of the distortion tensor is consistent with the usual definition as can be seen by calculating the spatial metric $g_{ij} = e^A_i e^B_j \delta_{AB} = \delta_{ij} + w_{ij} + w_{ji} + w^A_i w^B_j \delta_{AB}$ which matches the metric from elasticity theory [15]. The stress-energy response from Eq. (7) is

$$\langle T^\mu_a\rangle = \zeta_{\rm reg}\eta_{ab}\epsilon^{\mu\nu\rho}\partial_\nu e^b_\rho. \tag{19}$$

Since e^0_μ does not enter, this simplifies to a momentum-density $\langle T^0_A\rangle = \zeta_{\rm reg}\eta_{AB}\epsilon^{ij}\partial_i e^B_j$ and a momentum-current $\langle T^i_A\rangle = \zeta_{\rm reg}\eta_{AB}\epsilon^{i\nu\rho}\partial_\nu e^B_\rho$. These satisfy the continuity equations $\partial_t\langle T^0_A\rangle = -\partial_i\langle T^i_A\rangle$. Restricting ourselves to linear elasticity theory we can freely switch between frame (a) and local coordinate (μ) indices in the response equations. Thus $\langle T^\mu_a\rangle = \langle T^\mu_\nu\rangle + O((\delta e)^2)$. Also, spacetime indices are raised or lowered using the unperturbed metric.

For $\langle T^0_a\rangle \neq 0$, u^A cannot be single valued: it is a dislocation with Burgers vector b^A at $\mathbf{x}_0$, for which $\epsilon^{ij}\partial_i w^A_j = \epsilon^{ij}\partial_i e^A_j = -b^A\delta^{(2)}(\mathbf{x} - \mathbf{x}_0)$. Thus, the momentum-density response simplifies to $\langle T_{0j}\rangle = -\zeta_{\rm reg}\sum_m b^{(m)}_j\delta^{(2)}(\mathbf{x} - \mathbf{x}_m)$ where the $\mathbf{x}_m$ are the locations of dislocations and $b^{(m)}_j$ is the Burgers vector of the mth dislocation. For a lattice system, the dislocation is the fundamental quantized unit of torsion since transporting an electron around a defect translates the wave function by a multiple of a lattice constant. An edge dislocation with $|\mathbf{b}| = a$ (where a is the lattice constant) contains a momentum density of $\frac{\hbar}{8\pi\ell}\frac{a}{\ell}$ along the direction of $\mathbf{b}$. To think about smooth torsion deformations we need to take the continuum limit and deformations are a continuous distribution of dislocations. As illustrated in Fig. 1(a), this response is a momentum density bound to a torsion "flux" analogous to charge density bound to an EM flux in the bulk of a CI. Note that Fig. 1 is heuristic, since a realistic edge dislocation is not simply a cut into the material.

The physical interpretation of the momentum-current response $\langle T_{ij}\rangle$ is not as simple because it is more difficult to picture a torsion electric field. In the 2D plane, a moving dislocation (torsion flux) will generate a torsion electric field via the analog Faraday effect. Since we have seen that dislocations naturally carry a momentum density, moving it will generate a momentum-current density as per the response equation. In fact, the momentum current due to the moving dislocation is being carried perpendicular to the induced torsion electric field.

Another realization of the momentum-current response is obtained by using another instance of the Faraday effect: roll the CI into a cylinder and then insert a torsion flux into the cylindrical hole. This can be thought of as threading a dislocation into the hole of the cylinder so that any particles traveling around the hole will be translated by the Burgers vector b_a of the threaded dislocation. Changing b_a

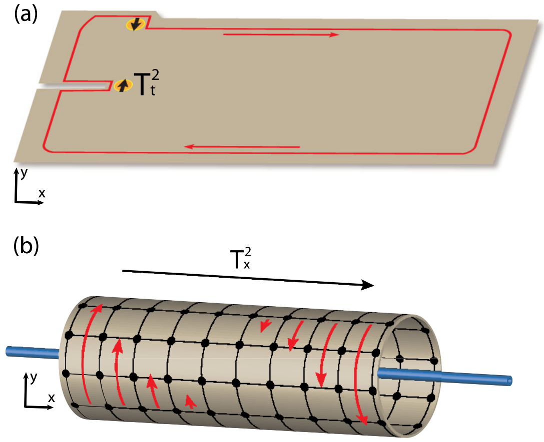

FIG. 1 (color online). (a) Chern insulator deformed by a dislocation-antidislocation pair, separated in the y direction. For each dislocation, the momentum density is in the direction of the Burgers vector. (b) Chern insulator on a cylinder with a (nonuniform) dislocation threading the cylinder. Local displacements are shown by red arrows. This gives rise to a momentum-current response along the cylinder which carries a momentum component parallel to the Burgers vector of the threaded dislocation, i.e., parallel to the red arrows.

as a function of time creates a torsion electric field the same way that a changing magnetic flux causes a circulating electric field. One key difference with the EM case is the necessity to preserve the total area to isolate torsion effects as shown in Fig. 1(b). Thus, one natural experiment is a torque experiment where a cylinder of CI is twisted. This is equivalent to threading a dislocation with a position dependent Burgers vector.

The formalism developed here is a natural generalization of classical elasticity theory. If $w^a_\mu = 0$ on the boundary, we can rewrite the effective Lagrangian [Eq. (7)] as

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \frac{\zeta_{\text{reg}}}{2}\epsilon^{\mu\nu\rho}\eta_{ab}w^a_\mu\partial_\nu w^b_\rho \\ &= \frac{\zeta_{\text{reg}}}{2}\epsilon^{\mu\nu\rho}\eta^{\sigma\lambda}(u_{\mu\sigma}\partial_\nu u_{\rho\lambda} + M_{\mu\sigma}\partial_\nu M_{\rho\lambda} \\ &\quad + 2M_{\mu\sigma}\partial_\nu u_{\rho\lambda}),\\ u_{\mu\nu} &= \frac{1}{2}\left(\frac{\partial u_\mu}{\partial x_\nu} + \frac{\partial u_\nu}{\partial x_\mu}\right), \qquad M_{\mu\nu} = \frac{1}{2}\left(\frac{\partial u_\mu}{\partial x_\nu} - \frac{\partial u_\nu}{\partial x_\mu}\right)\end{aligned}$$

within linear elasticity theory; $u_{\mu\nu}$ and $M_{\mu\nu}$ are the strain and rotation tensors, respectively. The first term is the torsional viscosity which is the Lorentz invariant version of the QH viscosity [5]. The stress-energy tensor response $\langle T_{\mu\nu}\rangle$ is not necessarily symmetric and thus does not fit in classical elasticity (independent of $M_{\mu\nu}$). It is natural to interpret the stress response within micropolar (Cosserat) elasticity theory which takes the local rotational degrees of freedom of the medium into account [16,17]. The distinction here is clear since Dirac fermions couple directly to the triad and not the metric, and the spinor nature of the Dirac equation gives local rotational degrees of freedom.

Finally, we briefly mention two interesting 3D generalizations, the details of which will be presented elsewhere. The first is an anisotropic extension to 3D with the form

$$S_{\text{eff}}[e^a_\mu] = \frac{\zeta_\mu}{2}\int d^4x\epsilon^{\mu\nu\rho\sigma}e^a_\nu\partial_\rho e^b_\sigma\eta_{ab} \qquad (20)$$

where ζ_μ is a vector of viscosity coefficients which is analogous to the 3D IQHE [18]. IQHE or QAHE states which are "stacked" along a direction perpendicular to the vector ζ_μ exhibit the viscosity response in Eq. (20). This action is basically equivalent to the 3D viscosity response of Ref. [9]. For a 3D strong TI we find

$$S_{\text{eff}}[e^a_\mu] = \frac{1}{2}\int d^4x\zeta^{(3)}\epsilon^{\mu\nu\rho\sigma}\partial_\mu e^a_\nu\partial_\rho e^b_\sigma\eta_{ab} \qquad (21)$$

which is a total derivative unless $\zeta^{(3)}$ is not a constant. Hence, on the surface of a 3DTI (where $\zeta^{(3)}$ has a domain wall) there will be a dissipationless viscoelastic response. This is expected since the surface also contains a QHE. In gravity theories with torsion the isotropic term is known as the Nieh-Yan term [19].

We leave open the question on how to experimentally measure this response in two-dimensional electron gases or TI's. First, unlike electric charge, momentum is not conserved when translation symmetry is broken, as it is in any realistic material. Additionally, our result seems to be generically nonquantized and somewhat regularization dependent. The reason these issues do not appear in the quantum Hall calculations is because the kinetic energy is quenched and each single-particle state contributes the same amount to the viscosity. We strongly believe this would be modified if one considers a lattice with a uniform magnetic field (Hofstadter problem) instead of a continuum Hamiltonian. Because the Hall viscosity is a mix of a geometric response with some topological flavor, it will have some nonuniversal features in any realistic system. With translation and rotation symmetry the viscosity was shown to be quantized [9], but the only physical response that has been linked to viscosity, and is insensitive to rotational invariance is the edge-dipole moment [8]. However, the latter result only applies to unreconstructed edges and is sure to be modified with real edge theories. The connection between the quantitative value of the viscosity and real experiments, as well as the bulk-edge correspondence for generic edge theories is not well understood and remains an open question.

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