New Standing Solitary Waves in Water

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(Received 5 October 2010; published 6 July 2011)

By means of the parametric excitation of water waves in a Hele-Shaw cell, we report the existence of two new types of highly localized, standing surface waves of large amplitude. They are, respectively, of odd and even symmetry. Both standing waves oscillate subharmonically with the forcing frequency. The two-dimensional even pattern presents a certain similarity in the shape with the 3D axisymmetric oscillon originally recognized at the surface of a vertically vibrated layer of brass beads. The stable, 2D odd standing wave has never been observed before in any media.

DOI: 10.1103/PhysRevLett.107.024502 PACS numbers: 47.54.−r, 47.35.Fg

Parametrically driven surface waves, known as Faraday waves, can be excited at the free surface of a fluid when the latter is submitted to periodical vertical oscillations, provided that the input energy is large enough to balance the dissipation due to the fluid viscosity [1–3]. At the laboratory scale, this experiment is a privileged way to explore some nonlinear properties of water waves.

Numerous experimental studies evidenced the appearance of various spatial patterns for 3D surface waves, which can be interpreted as nonlinear couplings between waves of different wavelengths. Depending on the driving amplitude and frequency, the observed patterns can be stripes, squares, hexagons [4,5], and even quasicrystalline point symmetries (with loss of translational symmetries) [6,7]. The classical theoretical framework explaining the emergence of these patterns relies on the devising of an amplitude equation aimed at capturing basic symmetries and gauge invariances while keeping the nonlinearities of lowest degrees [4,8–13].

In the present Letter, we report the existence of two 2D highly localized standing patterns, respectively, of even and of odd symmetry. They are obtained by vibrating vertically a Hele-Shaw (i.e., nearly two-dimensional) cell, partly filled with water. Both patterns oscillate subharmonically with the forcing vibration, the latter being purely sinusoidal with a single frequency. The profile of the 2D even pattern resembles somehow an axial slice of the 3D axisymmetric oscillon obtained with brass beads [14]. On the other hand, the existence of an oscillon of odd parity had never been reported in any media up to now.

The system studied is a fluid layer about 5 cm deep confined in a vertical glass cell (1.7 mm breadth, 30 cm long). The liquid is distilled water. The temperature is regulated within a precision of 0.1 °C in order to reduce fluctuations in surface tension and viscosity. The fluid vessel is mounted on a shaker and experiences a vertical purely sinusoidal motion (see Fig. 1). The amplitude of the cell oscillations can be driven up to 20 mm. The driving frequency is monitored with a synthesizer and is stable to about 0.1%. The surface deformation is recorded by means of a fast camera (250 fps) positioned perpendicularly to the cell. For water and for the explored range of vibration frequencies, the thickness of the cell corresponds to a Hele-Shaw configuration; i.e., the gap ℓ between the two lateral vertical glasses is small compared to the observed characteristic length of the wave. The advantage of the Hele-Shaw configuration is twofold. First, it increases considerably the domain of existence of oscillonic, localized waves (compared to the case of a 3D tank). Second, it justifies the assumption of an irrotational motion to treat theoretically the dynamics of parametrically forced water waves in the presence of viscous damping.

The experimental protocol is the following. We start from the at-rest equilibrium state. We choose a forcing frequency in the range 8–20 Hz (say, 10 Hz), and we increase slowly the oscillation amplitude up to the formation of spatially periodic standing waves. The Faraday waves appear with a finite amplitude at the acceleration threshold corresponding to the nondimensioned acceleration $F_\uparrow$ ($F_\uparrow = 2.24$ at 10 Hz), where $F = \Gamma/g$, Γ being the amplitude of the forcing acceleration and g being the

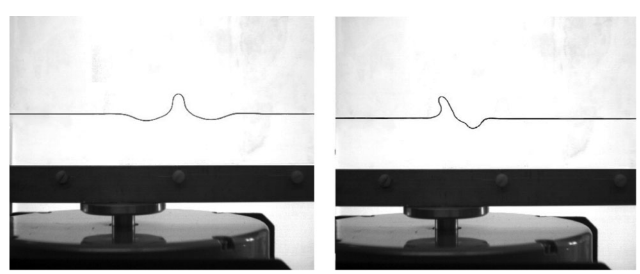

FIG. 1. Even (left) and odd (right) standing solitary waves. Driving frequency, 11 Hz; vibration amplitude, 4.1 mm; the wave amplitudes are of the order of 1.2 cm.

acceleration due to gravity. Next, the driving acceleration is decreased down to the vanishing of the surface waves. Another threshold $F_\downarrow$ is thus reported ($F_\downarrow = 1.13$ at 10 Hz). The Faraday instability is therefore clearly hysteretic (of subcritical type) for water in a Hele-Shaw configuration (see Fig. 2). Once the two thresholds are located, one starts from the motionless state at $F = 0$, and the vibration amplitude is then increased in order to attain the hysteretic region ($F_\downarrow < F < F_\uparrow$). According to the preparing process, the free surface remains flat. Next, the free surface is locally perturbed with an external probe that is quickly removed. Thus, we observe a standing, localized surface wave, oscillating with angular frequency $\omega = \Omega/2$ (Ω the forcing angular frequency), as shown in Fig. 1. For rigorously identical excitation parameters (F and Ω), the localized wave can be of various shapes, with either an even or an odd symmetry [Figs. 1(a) and 1(b); movies can be accessed in Ref. [15]]. Note that here we abusively use the term "odd," because crests and troughs are not exactly of the same magnitudes, as is the case with nonlinear water waves—see Figs. 1 and 3). The difference in the observed shapes may likely be attributed to the probe motion. It might be envisaged that the even wave could result from the coupling and binding of two odd localized standing waves. In Figs. 3(a) and 3(b), the time evolution of, respectively, the even and the odd patterns is reported. Note that, in both cases, an overturning of the free surface can be observed temporally.

As mentioned above, the profile of the even standing solitary wave observed here resembles somehow the profile of the 3D axisymmetric oscillon recognized in a vibrated layer of bronze spheres [14]. 3D axisymmetric oscillons have also been seen in liquids [16,17], but they displayed noticeable differences from the presently reported ones. In Ref. [16], the fluid is a non-Newtonian clay suspension, and the authors attribute the generation of oscillons to a nonlinear shear thinning with frequency-dependent viscosity. In Ref. [17], the fluid is a Newtonian silicon oil, and the parametric forcing is constituted with two commensurate angular frequencies $m\Omega$ and $n\Omega$ (where m and n are prime integers). The oscillon generated by these two superimposed frequencies vibrates *harmonically* with the basic forcing frequency Ω corresponding to the greatest common divisor of the two frequencies. On the other hand, in the present Hele-Shaw cell, both odd and even 2D oscillons oscillate *subharmonically*. Moreover, the present study is the first experimental one performed in a Hele-Shaw cell, and the even 2D localized standing wave that we report here is very different from the 3D axisymmetric oscillon. Indeed, the solutions of the wave equation and their stability differ drastically according to the space dimensionality. Furthermore, concerning the existence of the odd localized standing wave, to our knowledge, it had never been observed in any media hitherto.

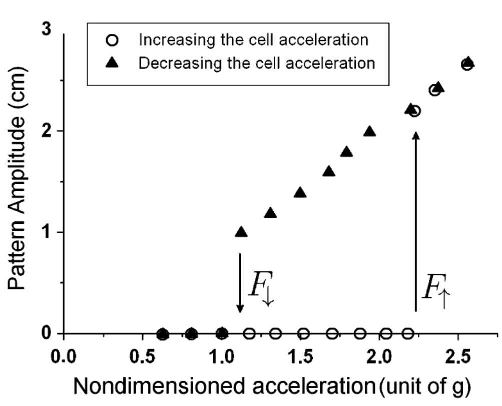

FIG. 2. Experimental determination of the region of bistability. The domain of bistability $F \in [F_\downarrow, F_\uparrow]$ corresponds to the region of existence of the standing solitary waves as shown in Fig. 1.

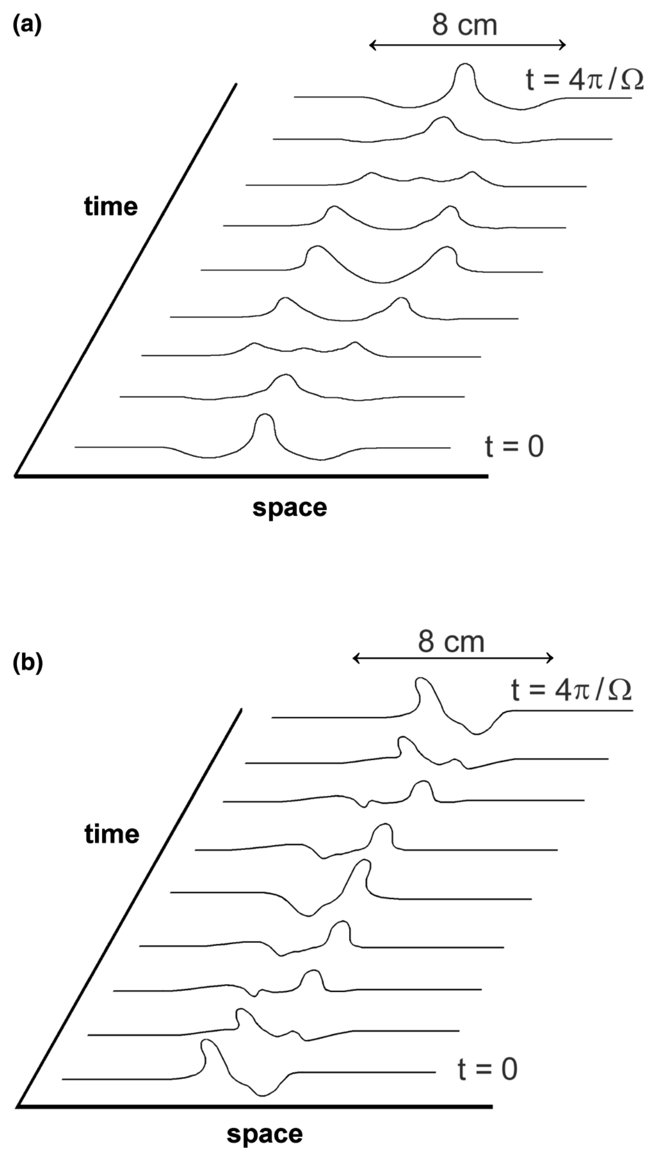

FIG. 3. Temporal evolution of the profile of the subharmonic solitary waves. (a) Even pattern. (b) Odd pattern.

Various types of amplitude equations—such as the forced dissipative nonlinear Schrödinger equation [18,19] and the complex Swift-Hohenberg equation [20,21]—have been designed to account for localized standing patterns. From these theoretical studies, it emerges that the two ingredients required for the formation of oscillons are mainly dissipation associated with hysteresis. We note in passing that amplitude equations are well suited to describe slowly modulated small-amplitude sinusoidal waves (e.g., envelope soliton), which is not the case for oscillons where the "carrier" wave and its "envelope" have comparable extents; moreover, the solitary standing waves reported here have large amplitudes (i.e., they are very steep waves). Nonetheless, despite their limitation, amplitude equations give some insight and are thus considered here.

In the Hele-Shaw configuration, it is convenient to introduce the gap-averaged velocity $\bar{\boldsymbol{u}}$ of the fluid (overbars denote quantities averaged over the gap). The (constant) gap width ℓ is small compared to the water depth h, the characteristic wavelength $2\pi/k$, the wave amplitude a, and the cell length. Thus, the average is performed by assuming that the velocity profile is parabolic across the gap (Poiseuille's flow). In the reference frame of the cell, the gap averaging of the Navier-Stokes equation yields the two-dimensional equation

$$\bar{\boldsymbol{u}}_t + \tfrac{6}{5}(\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}} = -\rho^{-1}\boldsymbol{\nabla}\bar{P} - \tilde{\boldsymbol{g}} - \gamma\bar{\boldsymbol{u}}, \quad (1)$$

where $\gamma = 12\nu/\ell^2$ is the so-called *Rayleigh's external viscosity* and $\tilde{\boldsymbol{g}} = (0, \tilde{g})$ with $\tilde{g} = g[1 - F\cos(\Omega t)]$ is the apparent gravity acceleration experienced by the liquid in the vibrated cell. The numerical factor $6/5 = \overline{\boldsymbol{u}^2}/\bar{\boldsymbol{u}}^2$ characterizes an increase in nonlinear effects due to the viscous boundary layers in the direction perpendicular to the cell.

In Eq. (1), the viscous diffusion term $\nu\boldsymbol{\nabla}^2\bar{\boldsymbol{u}}$ has been neglected because it is very small compared to $\gamma\bar{\boldsymbol{u}}$. More precisely, for the observed characteristic wave numbers k, the viscous damping time γ^{-1} imposed by the Hele-Shaw geometry is roughly 250 times shorter than the diffusion time $(\nu k^2)^{-1}$ arising from the term $\nu\boldsymbol{\nabla}^2\bar{\boldsymbol{u}}$. It is interesting to point out that, in the range corresponding to the oscillon appearance, the viscous damping time γ^{-1} ($\simeq 0.24$ s) imposed by the Hele-Shaw geometry compares with the response period π/Ω (typically 0.2 s). It can be easily seen that any vorticity field at the initial time is rapidly damped (in the absence of the source of vorticity) due to the dissipation term $\gamma\bar{\boldsymbol{u}}$. Since we are interested in established regimes, we can neglect the vorticity and hence consider potential flows. Thus, by introducing a velocity potential ϕ such that $\bar{\boldsymbol{u}} = \boldsymbol{\nabla}\phi$, Eq. (1) can be integrated into a Bernoulli equation, and the water wave equations to be investigated are

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{for } -h \leq y \leq \eta, \quad (2)$$

$$\phi_y = 0 \quad \text{at } y = -h, \quad (3)$$

$$\eta_t + \eta_x\phi_x - \phi_y = 0 \quad \text{at } y = \eta, \quad (4)$$

$$\phi_t + \tfrac{3}{5}(\boldsymbol{\nabla}\phi)^2 + \gamma\phi + \tilde{g}\eta = 0 \quad \text{at } y = \eta, \quad (5)$$

where x, y, and t are, respectively, the horizontal, upward vertical, and temporal variables, and $y = \eta(x, t)$, $y = 0$, and $y = -h$ are the equations of the impermeable free surface, of the mean water level, and of the impermeable horizontal bottom, respectively. The surface tension has been neglected in Eq. (5), since the capillary length (which is equal to 2.7 mm for water and air) is small compared to the observed wavelengths. The water wave equations thus consist in the incompressibility and irrotationality in the bulk, with the impermeability boundary conditions of the free surface and of the cell bottom, together with the zero pressure at the free surface.

We consider first the linear approximation of Eqs. (2)–(5). Introducing a damped Fourier-like integral representation for the surface in the form $\eta(x, t) = \int_{-\infty}^{\infty} \zeta(k, t)e^{ikx-\gamma t/2}dk$, we found after some algebra that ζ satisfies a damped Mathieu equation [22]

$$\zeta_{tt} + [\omega_0^2 - \tfrac{1}{4}\gamma^2 - \omega_0^2 F\cos(\Omega t)]\zeta = 0, \quad (6)$$

where $\omega_0^2 = gk\tanh(kh)$. It is well known that such systems exhibit resonance conditions for $n\Omega = 2\omega_0$, n being an integer. Equation (6) as been derived by Benjamin and Ursell [2] without dissipation ($\gamma = 0$). If dissipation is disregarded, the resonance conditions obviously correspond to unbounded solutions. Dissipation is often added empirically into the Mathieu equation via an *ad hoc* term. With the present derivation, dissipation is obtained directly from the exact equations (i.e., γ is known explicitly in terms of the physical parameters), thanks to the simplifications yielded by the Hele-Shaw configuration.

We investigate now the effect of nonlinearities. Hereafter, in order to simplify the algebra, the water depth h is considered as infinite and we consider only one spatial wave number k corresponding to a single standing mode. The set of equations (2)–(5) is then solved approximately via a perturbative scheme using a multiple scale expansion [23]. After some algebra, an approximate equation is obtained for the free surface written as $\eta(x, t) = \mathrm{Re}\{A(t)e^{i(\Omega/2)t-i(\pi/4)}\}\cos(kx) + O(A^2)$, where the complex amplitude $A(t)$ satisfies the equation

$$A_t = (-\alpha_1 + i\alpha_2)A + \alpha_3 A^* - i\alpha_4|A|^2 A, \quad (7)$$

an asterisk denoting the complex conjugate and where the α_n are real parameters, i.e., $\alpha_1 = \gamma/2$, $\alpha_2 = \omega_0 - \Omega/2$, $\alpha_3 = \omega_0 F/4$, and $\alpha_4 = \omega_0 k^2/10$ with $\omega_0^2 = gk$.

In Eq. (7), the coefficient α_1 is related to the viscous damping of the wave, the coefficient α_2 corresponds to the detuning between the natural frequency of the mode k in the limit of infinitesimal amplitudes and half the forcing frequency, the term involving A^* is related to the parametric forcing, and the last term corresponds to the nonlinear frequency shift with the wave amplitude. The linear and nonlinear stability analysis of an equation of the same form of Eq. (7) has been studied by Meron [24]. For $\Omega > 2\omega_0$, the instability leading to the formation of a standing wave is of the supercritical Hopf type, while it becomes subcritical for $\Omega < 2\omega_0$.

Seeking for a solution of constant amplitude $A = a\exp(i\delta)$, we obtain the dispersion relation $[1-(ka)^2/10-\Omega/2\omega_0]^2 = F^2/16 - \gamma^2/(4\omega_0^2)$. This relation can be satisfied only if the right-hand side is positive, implying the condition $F \geq F_\downarrow = 2\gamma/\omega_0$. A remarkable feature of Eq. (7) is that there is a domain of bistability, defined by $\Omega < 2\omega_0$, $F_\downarrow < F < F_\uparrow$ (with $F_\uparrow = \frac{4}{\omega_0}\sqrt{[\omega_0 - (\Omega/2)]^2 + (\gamma^2/4)}$), where both the flat surface state and the standing wave of angular frequency $\Omega/2$ are stable. However, Eq. (7) is unable to predict the exact shape of the free surface. This is not surprising, because this equation is derived under the assumption of small amplitude, while the standing solitary waves we report here are *extreme* waves for which there are no small parameters; moreover, these waves exist only with a finite amplitude (i.e., they do not bifurcate from the rest). Thus, the design of a highly nonlinear theory would be necessary to account for the reported solitary waves, but it is still far beyond the existing methods.

As previously suggested in Refs. [25,26], the localized patterns that we observed can be interpreted as resulting from the coexistence in space of both flat and wavy regions. The simultaneous existence of these flat and wavy states as results of the same forcing is therefore conditioned by the bistability phenomenon accounted for above. Thus, as advanced by Umbanhowar, Melo, and Swinney [14], hysteresis and dissipation are essential ingredients. Analog mechanisms leading to subcritical Hopf bifurcations and giving rise to localized patterns are also encountered in many other fields, such as nonlinear optics [27], chemistry [28], and biology [29].

We acknowledge Alain Pumir and Nail Akhmediev for fruitful discussions. This project has been partially supported by CNRS and Région PACA.

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