Gravitational Anomaly and Transport Phenomena

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Quantum anomalies give rise to new transport phenomena. In particular, a magnetic field can induce an anomalous current via the chiral magnetic effect and a vortex in the relativistic fluid can also induce a current via the chiral vortical effect. The related transport coefficients can be calculated via Kubo formulas. We evaluate the Kubo formula for the anomalous vortical conductivity at weak coupling and show that it receives contributions proportional to the gravitational anomaly coefficient. The gravitational anomaly gives rise to an anomalous vortical effect even for an uncharged fluid.

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Introduction.—The effects quantum anomalies have on the hydrodynamics of otherwise conserved currents have recently received much attention. Two such effects are known: an anomalous magnetic effect [1] and an anomalous vortical effect [2,3]. In the first an (external) magnetic field induces a current via the axial anomaly, whereas the second is the generation of a current due to the presence of a vortex in the charged relativistic fluid. These effects have been argued to lead to observable event by event parity violation and a charge separation effect in noncentral heavy ion collision at RHIC and LHC [4]. In the hydrodynamic constitutive relations these effects lead to the existence of a new class of transport coefficients.

A first principle calculation of transport coefficients is possible via Kubo formulas. The Kubo formula for the anomalous magnetic conductivity has been derived and applied in [5] whereas the one for the anomalous vortical conductivity has been established recently in [6]. They are

$$\sigma^{\mathcal{B}}_{AB} = \lim_{k_n\to 0}\sum_{ij}\varepsilon_{ijn}\frac{-i}{2k_n}\langle J^i_A J^j_B\rangle|_{\omega=0}, \tag{1}$$

$$\sigma^{\mathcal{V}}_{A} = \lim_{k_n\to 0}\sum_{ij}\varepsilon_{ijn}\frac{-i}{2k_n}\langle J^i_A T^{0j}\rangle|_{\omega=0}, \tag{2}$$

where J^μ_A are the (anomalous) currents and $T^{\mu\nu}$ is the energy momentum tensor. These Kubo formulas allow the calculation of the transport coefficients in the constitutive relations

$$J^i_A = \sigma^{\mathcal{B}}_{AB}\varepsilon^{ijk}\partial_j\mathcal{A}^B_k + \sigma^{\mathcal{V}}_A\varepsilon^{ijk}\partial_j v_k, \tag{3}$$

where $\mathcal{A}^B_k$ are the spatial components of a collection of gauge fields and v_k is the local fluid velocity. See [6] for a discussion. We note that we can substitute the fluid velocities by the gravitomagnetic potential. To do so we go to the rest frame of the fluid defined by $u^\mu = (1,0,0,0)$ but switch on a gravitomagnetic field in the metric according to

$$ds^2 = dt^2 + 2\vec{\mathcal{A}}_g d\vec{x}dt - d\vec{x}^2. \tag{4}$$

Using this metric we can compute the local fluid velocity $u_\mu = (1, \vec{\mathcal{A}}_g)$ such that $\vec{v} = \vec{\mathcal{A}}_g$. All these expressions are valid only up to first order in the external fields $\mathcal{A}^A_k, v_k$. Plugging this into the constitutive relation (3) and noting that $\mathcal{A}^i_g$ sources T^{0i} leads to the Kubo formula for the vortical conductivity.

We will now evaluate the Kubo formulas (1) and (2) for a theory of N free right-handed fermions Ψ^f transforming under a global symmetry group G generated by matrices $(T_A)^f{}_g$. We denote the generators in the Cartan subalgebra by H_A. Chemical potentials μ_A can be switched on only in the Cartan subalgebra. Furthermore, the presence of the chemical potentials breaks the group G to a subgroup $\hat{G}$. Only the currents that lie in the unbroken subgroup are conserved (up to anomalies) and participate in the hydrodynamics. The chemical potential for the fermion Ψ^f is given by $\mu^f = \sum_A q^f_A\mu_A$, where we write the Cartan generator $H_A = q^f_A\delta^f{}_g$ in terms of its eigenvalues, the charges q^f_A. The unbroken symmetry group $\hat{G}$ is generated by the matrices $T^f_{A\,g}$ fulfilling $T^f_{A\,g}\mu^g = \mu^f T^f_{A\,g}$. There is no summation over indices in the last expression. From now on we will assume that all currents $\vec{J}_A$ lie in directions indicated in the preceding equation. We define the chemical potential through boundary conditions on the fermion fields around the thermal circle [7], $\Psi^f(\tau) = -e^{\beta\mu^f}\Psi^f(\tau-\beta)$ with $\beta = 1/T$. Therefore the eigenvalues of ∂_τ are $i\tilde{\omega}_n + \mu^f$ for the fermion species f with $\tilde{\omega}_n = \pi T(2n+1)$ the fermionic Matsubara frequencies. A convenient way of expressing the currents is in terms of Dirac fermions and writing

$$J^i_A = \sum_{f,g=1}^{N} T^g_{A\,f}\bar{\Psi}_g\gamma^i\mathcal{P}_+\Psi^f, \tag{5}$$

$$T^{0i} = \frac{i}{2}\sum_{f=1}^{N}\bar{\Psi}_f(\gamma^0\partial^i + \gamma^i\partial^0)\mathcal{P}_+\Psi^f, \tag{6}$$

where we used the chiral projector $\mathcal{P}_\pm = \frac{1}{2}(1\pm\gamma_5)$. The fermion propagator is

$$S(q)^f{}_g = \frac{\delta^f{}_g}{2}\sum_{t=\pm}\Delta_t(i\tilde{\omega}^f, \vec{q})\mathcal{P}_+\gamma_\mu \hat{q}_t^\mu, \quad (7)$$

$$\Delta_t(i\tilde{\omega}^f, q) = \frac{1}{i\tilde{\omega}^f - tE_q}, \quad (8)$$

with $i\tilde{\omega}^f = i\tilde{\omega}_n + \mu^f$, $\hat{q}_t^\mu = (1, t\hat{q})$, $\hat{q} = \frac{\vec{q}}{E_q}$ and $E_q = |\vec{q}|$. We can easily include left-handed fermions as well. They contribute in all our calculations in the same way as the right-handed ones up to a relative minus sign.

Evaluation of Kubo formulas.—We will address in detail the computation of the vortical conductivities Eq. (2) and sketch only the calculation of the magnetic conductivities since the latter one is a trivial extension of the calculation of the chiral magnetic conductivity in [5].

Vortical conductivity.—The vortical conductivity is defined from the retarded correlation function of the current $J_A^i(x)$ (5), and the energy momentum tensor or energy current $T^{0j}(x')$ (6), i.e.,

$$G_A^{\mathcal{V}}(x - x') = \tfrac{1}{2}\epsilon_{ijn} i\theta(t - t')\langle[J_A^i(x), T^{0j}(x')]\rangle. \quad (9)$$

Going to Fourier space, one can evaluate this quantity as

$$G_A^{\mathcal{V}}(k) = \frac{1}{4}\sum_{f=1}^{N} T_{Af}^f \frac{1}{\beta}\sum_{\tilde{\omega}^f}\int\frac{d^3q}{(2\pi)^3}\epsilon_{ijn}\mathrm{tr}[S^f{}_f(q)\gamma^i S^f{}_f(q+k) \times(\gamma^0 q^j + \gamma^j i\tilde{\omega}^f)]. \quad (10)$$

The vertex of the two quarks with the graviton is $\sim\delta^f{}_g$, and therefore we find only contributions from the diagonal part of the group $\hat{G}$. Our metric is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. We can split $G_A^{\mathcal{V}}$ into two contributions

$$G_A^{\mathcal{V}}(k) = G_{A,(0j)}^{\mathcal{V}}(k) + G_{A,(j0)}^{\mathcal{V}}(k), \quad (11)$$

which correspond to the terms $\gamma^0 q^j$ and $\gamma^j i\tilde{\omega}^f$ in Eq. (10), respectively. We focus first on the computation of $G_{A,(0j)}^{\mathcal{V}}$. The integrand of Eq. (10) for $G_{A,(0j)}^{\mathcal{V}}$ can be written as

$$I_{A,(0j)}^{\mathcal{V}} = \frac{1}{4}q^j\sum_{t,u=\pm}\epsilon_{ijn}\mathrm{tr}[\gamma_\mu\gamma^i\gamma_\nu\gamma^0\mathcal{P}_+]\Delta_t(i\tilde{\omega}^f, \vec{q})\Delta_u(i\tilde{\omega}^f + i\omega_n, \vec{q} + \vec{k})\hat{q}_t^\mu(\widehat{q+k})_u^\nu. \quad (12)$$

From a computation of the Dirac trace in Eq. (12) one has two contributions

$$\epsilon_{ijn}\mathrm{tr}[\gamma_\mu\gamma^i\gamma_\nu\gamma^0]a^\mu b^\nu = 4\epsilon_{ijn}(a^i b^0 + a^0 b^i), \quad (13)$$

$$\epsilon_{ijn}\mathrm{tr}[\gamma_\mu\gamma^i\gamma_\nu\gamma^0\gamma_5]a^\mu b^\nu = 4i(a_j b_n - a_n b_j). \quad (14)$$

Using Eqs. (12)–(14) one can express $G_{A,(0j)}^{\mathcal{V}}(k)$ as

$$G_{A,(0j)}^{\mathcal{V}}(k) = \frac{1}{8}\sum_{f=1}^{N} T_{Af}^f\frac{1}{\beta}\sum_{\tilde{\omega}^f}\int\frac{d^3q}{(2\pi)^3}q^j \times\sum_{t,u=\pm}\bigg[\epsilon_{ijn}\bigg(t\frac{q^i}{E_q} + u\frac{k^i + q^i}{E_{q+k}}\bigg) + i\frac{tu}{E_qE_{q+k}}(q_jk_n - q_nk_j)\bigg]\Delta_t(i\tilde{\omega}^f, \vec{k}) \times\Delta_u(i\tilde{\omega}^f + i\omega_n, \vec{q} + \vec{k}). \quad (15)$$

Note that due to the antisymmetric tensor ϵ_{ijn}, the two terms proportional to q^i inside the bracket in Eq. (15) vanish. Regarding the term $\epsilon_{ijn}q^jk^i$, it leads to a contribution $\sim\epsilon_{ijn}k^jk^i$ after integration in d^3q, which is zero. Then the only term which remains is the one not involving ϵ_{ijn}. We can now perform the sum over fermionic Matsubara frequencies. One has

$$\frac{1}{\beta}\sum_{\tilde{\omega}^f}\Delta_t(i\tilde{\omega}^f, \vec{q})\Delta_u(i\tilde{\omega}^f + i\omega_n, \vec{q} + \vec{p}) = \frac{tn(E_q - t\mu^f) - un(E_{q+k} - u\mu^f) + \frac{1}{2}(u - t)}{i\omega_n + tE_q - uE_{q+k}}, \quad (16)$$

where $n(x) = 1/(e^{\beta x} + 1)$ is the Fermi-Dirac distribution function. In Eq. (16) we have considered that $\omega_n = 2\pi Tn$ is a bosonic Matsubara frequency [5]. After doing the analytic continuation, which amounts to replacing $i\omega_n$ by $k_0 + i\varepsilon$ in Eq. (16), one gets

$$G_{A,(0j)}^{\mathcal{V}}(k) = -\frac{i}{8}\sum_{f=1}^{N} T_{Af}^f\int\frac{d^3q}{(2\pi)^3}\frac{\vec{q}^2k_n - (\vec{q}\cdot\vec{k})q_n}{E_qE_{q+k}} \times\sum_{t,u=\pm}\frac{un(E_q - t\mu^f) - tn(E_{q+k} - u\mu^f) + \frac{1}{2}(t - u)}{k_0 + i\varepsilon + tE_q - uE_{q+k}}. \quad (17)$$

The term proportional to $\sim\frac{1}{2}(t - u)$ corresponds to the vacuum contribution, and it is ultraviolet divergent. By removing this term the finite temperature and chemical potential behavior is not affected, and the result becomes ultraviolet finite because the Fermi-Dirac distribution function exponentially suppresses high momenta. By making both the change of variable $\vec{q}\to -\vec{q} - \vec{k}$ and the interchange $u\to -t$ and $t\to -u$ in the part of the integrand involving the term $-tn(E_{q+k} - u\mu^f)$, one can express the vacuum subtracted contribution of Eq. (17) as

$$\hat{G}_{A,(0j)}^{\mathcal{V}}(k) = \frac{i}{8}k_n\sum_{f=1}^{N} T_{Af}^f\int\frac{d^3q}{(2\pi)^3}\frac{1}{E_qE_{q+k}}\bigg(\vec{q}^2 - \frac{(\vec{q}\cdot\vec{k})^2}{\vec{k}^2}\bigg) \times\sum_{t,u=\pm}u\frac{n(E_q - \mu^f) + n(E_q + \mu^f)}{k_0 + i\varepsilon + tE_q + uE_{q+k}}. \quad (18)$$

The result has to be proportional to k_n, so to reach this expression we have replaced q_n by $(\vec{q}\cdot\vec{k})k_n/\vec{k}^2$ in Eq. (17).

At this point one can perform the sum over u and the integration over angles. Then one gets the final result

$$\hat{G}^{\mathcal{V}}_{A,(0j)}(k) = \frac{i}{16\pi^2}\frac{k_n}{k^2}(k^2-k_0^2)\sum_{f=1}^{N} T^f_{Af}\int_0^\infty dq q f^{\mathcal{V}}(q) \times\left[1+\frac{1}{8qk}\sum_{t=\pm}[k_0^2-k^2+4q(q+tk_0)] \times\log\left(\frac{\Omega_t^2-(q+k)^2}{\Omega_t^2-(q-k)^2}\right)\right], \tag{19}$$

where $\Omega_t = k_0 + i\varepsilon + tE_q$, and $f^{\mathcal{V}}(q) = n(E_q - \mu^f) + n(E_q + \mu^f)$. The steps to compute $G^{\mathcal{V}}_{A,(j0)}$ in Eq. (11) are similar. In this case the Dirac trace leads to a different tensor structure, in which the only contribution comes from the trace involving γ_5, i.e., $\varepsilon_{ijn}\mathrm{tr}[\gamma_\mu\gamma^i\gamma_\nu\gamma^j\gamma_5]a^\mu b^\nu = 8i(a_n b_0 - a_0 b_n)$. The sum over fermionic Matsubara frequencies involves an extra $i\tilde{\omega}^f$. Following the same procedure as explained above, the vacuum subtracted contribution writes

$$\hat{G}^{\mathcal{V}}_{A,(j0)}(k) = -\frac{i}{32\pi^2}\frac{k_n}{k^3}\sum_{f=1}^{N} T^f_{Af}\int_0^\infty dq\sum_{t=\pm} f_t^{\mathcal{V}}(q,k_0) \times\left[4tqkk_0-(k^2-k_0^2)(2q+tk_0) \times\log\left(\frac{\Omega_t^2-(q+k)^2}{\Omega_t^2-(q-k)^2}\right)\right], \tag{20}$$

where $f_t^{\mathcal{V}}(q,k_0) = qf^{\mathcal{V}}(q) + tk_0 n(E_q + t\mu^f)$. The result for $\hat{G}^{\mathcal{V}}_A(k)$ writes as a sum of Eqs. (19) and (20), according to Eq. (11). From these expressions one can compute the zero frequency, zero momentum, limit. Since

$$\lim_{k\to 0}\lim_{k_0\to 0}\sum_{t=\pm}\log\left(\frac{\Omega_t^2-(q+k)^2}{\Omega_t^2-(q-k)^2}\right) = \frac{2k}{q}, \tag{21}$$

the relevant integrals are

$$\int_0^\infty dq q f^{\mathcal{V}}(q) = \int_0^\infty dq f_t^{\mathcal{V}}(q, k_0 = 0) = \frac{(\mu^f)^2}{2} + \frac{\pi^2}{6}T^2. \tag{22}$$

Finally it follows from Eqs. (19) and (20) that the zero frequency, zero momentum, vortical conductivity writes

$$\sigma^{\mathcal{V}}_A = \frac{1}{8\pi^2}\sum_{f=1}^{N} T^f_{Af}\left[(\mu^f)^2 + \frac{\pi^2}{3}T^2\right] = \frac{1}{16\pi^2}\left[\sum_{B,C}\mathrm{tr}(T_A\{H_B,H_C\})\mu_B\mu_C + \frac{2\pi^2}{3}T^2\mathrm{tr}(T_A)\right]. \tag{23}$$

Both $\hat{G}^{\mathcal{V}}_{A,(0j)}$ and $\hat{G}^{\mathcal{V}}_{A,(j0)}$ lead to the same contribution in $\sigma^{\mathcal{V}}_A$. Equation (23) constitutes our main result. The term involving the chemical potentials is induced by the chiral anomaly. More interesting is the term $\sim T^2$ which is proportional to the gravitational anomaly [8] as we will show in the discussion section. The Matsubara frequencies $\tilde{\omega}_n = \pi T(2n+1)$ generate a dependence on πT in the final result as compared to the chemical potentials, and then no factors of π show up for the term $\sim T^2$ in Eq. (23). Left-handed fermions contribute in the same way but with a relative minus sign. Left and right-handed fermions do not mix.

If instead of having taken the zero momentum limit at zero frequency, one took the zero frequency limit at zero momentum, the result would be $1/3$ of the result quoted in Eq. (23). The same factor appears in the magnetic conductivity when one interchanges the two limits [5].

Magnetic conductivity.—The magnetic conductivity in the case of a vector and an axial $U(1)$ symmetry was computed at weak coupling in [5]. Following the same method, we have computed it for the unbroken (non-Abelian) symmetry group $\hat{G}$. The relevant Green function is

$$G^{\mathcal{B}}_{AB} = \frac{1}{2}\sum_{f,g} T^g_{Af}T^f_{Bg}\frac{1}{\beta}\sum_{\tilde{\omega}^f}\int\frac{d^3q}{(2\pi)^3}\varepsilon_{ijn} \times\mathrm{tr}[S^f{}_f(q)\gamma^i S^f{}_f(q+k)\gamma^j]. \tag{24}$$

The evaluation of this expression is exactly as in [5] so we skip the details. The result is

$$\sigma^{\mathcal{B}}_{AB} = \frac{1}{4\pi^2}\sum_{f,g=1}^{N} T^f_{Ag}T^g_{Bf}\mu^f = \frac{1}{8\pi^2}\sum_C \mathrm{tr}(T_A\{T_B,H_C\})\mu_C. \tag{25}$$

No contribution proportional to the gravitational anomaly coefficient is found in this case.

Discussion.—In vacuum the anomaly appears on the level of three point functions. In the presence of external sources for the energy momentum tensor and the currents this is conveniently expressed through [8]

$$\nabla_\mu J^\mu_A = \varepsilon^{\mu\nu\rho\lambda}\left(\frac{d_{ABC}}{32\pi^2}F^B_{\mu\nu}F^C_{\rho\lambda} + \frac{b_A}{768\pi^2}\mathcal{R}^{\alpha}{}_{\beta\mu\nu}\mathcal{R}^{\beta}{}_{\alpha\rho\lambda}\right). \tag{26}$$

The anomaly coefficients are defined by

$$d_{ABC} = \tfrac{1}{2}[\mathrm{tr}(T_A\{T_B,T_C\})_R - \mathrm{tr}(T_A\{T_B,T_C\})_L], \tag{27}$$

$$b_A = \mathrm{tr}(T_A)_R - \mathrm{tr}(T_A)_L, \tag{28}$$

where the subscripts R, L stand for the contributions of right-handed and left-handed fermions.

We have computed the magnetic and vortical conductivity at weak coupling and we find contributions that are proportional to the anomaly coefficients (27) and (28). Nonzero values of these coefficients are a necessary and sufficient condition for the presence of anomalies [9]. Therefore the nonvanishing values of the transport coefficients (1) and (2) have to be attributed to the presence of chiral and gravitational anomalies.

Since the gravitational anomaly is fourth order in derivatives it is a bit surprising to find it contributing to first order transport coefficients. One possible intuitive explanation one could think of is that the gravitational field in the presence of matter gives rise to a fluid velocity u^μ, e.g., through frame dragging effects, and that this might effectively reduce the number of derivatives that enter in the hydrodynamic expansion.

The holographic calculation in AdS/CFT [6] did not show a contribution proportional to T^2. This is not surprising since only a holographic gauge Chern-Simons term was included. Holographic modeling of the gravitational anomaly calls, however, also for inclusion of a mixed gauge-gravitational Chern-Simons term of the form $A \wedge R \wedge R$ [10].

We find a nonvanishing vortical conductivity proportional to $\sim T^2$ even in an uncharged fluid. In [11] similar terms in the vortical conductivities have been argued for as undetermined integration constants without any relation to the gravitational anomaly.

It is also interesting to specialize our results to the case of one vector and one axial current with chemical potentials $\mu_R = \mu + \mu_A$, $\mu_L = \mu - \mu_A$, charges $q^R_{V,A} = (1, 1)$ and $q^L_{V,A} = (1, -1)$ for one right-handed and one left-handed fermion. We find (for a vector magnetic field)

$$\sigma^{\mathcal{B}}_{VV} = \frac{\mu_A}{2\pi^2}, \qquad \sigma^{\mathcal{B}}_{AV} = \frac{\mu}{2\pi^2}, \qquad \sigma^{\mathcal{V}}_{V} = \frac{\mu\mu_A}{2\pi^2}, \qquad \sigma^{\mathcal{V}}_{A} = \frac{\mu^2 + \mu_A^2}{4\pi^2} + \frac{T^2}{12}. \tag{29}$$

Here $\sigma^{\mathcal{B}}_{VV}$ is the chiral magnetic conductivity [5], $\sigma^{\mathcal{B}}_{AV}$ describes the generation of an axial current due to a vector magnetic field [12], $\sigma^{\mathcal{V}}_{V}$ is the vector vortical conductivity, $\sigma^{\mathcal{V}}_{A}$ is the axial vortical conductivity and the only one sensitive to the gravitational anomaly.

In [13] enhanced production of high spin hadrons (especially Ω^- baryons) perpendicular to the reaction plane in heavy ion collisions has been proposed as an observational signature for the chiral separation effect. Three sources of chiral separation have been identified: the anomaly in vacuum, the magnetic and the vortical conductivities of the axial current J^μ_A. Of these the contribution of the vortical effect was judged to be subleading by a relative factor of 10^{-4}. The T^2 term in (29) leads, however, to a significant enhancement. If we take μ to be the baryon chemical potential $\mu \approx 10$ MeV, neglect μ_A as in [13] and take a typical RHIC temperature of $T = 350$ MeV, we see that the temperature enhances the axial chiral vortical conductivity by a factor of the order of 10^4. We expect the enhancement at the LHC to be even higher due to the higher temperature.

Beyond applications to heavy ion collisions leading to charge and chiral separation effects [13,14] it is tempting to speculate that the new terms in the chiral vortical conductivity might play a role in the early Universe. Indeed it has been suggested before that the gravitational anomaly might give rise to Lepton number generation, e.g., in [15]. The lepton number separation due to the gravitational anomaly could contribute to generate regions with nonvanishing lepton number.

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[1] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[2] D. Kharzeev and A. Zhitnitsky, Nucl. Phys. **A797**, 67 (2007).
[3] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, J. High Energy Phys. 01 (2009) 055; N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, and P. Surowka, J. High Energy Phys. 01 (2011) 094; D. T. Son and P. Surowka, Phys. Rev. Lett. **103**, 191601 (2009).
[4] B. I. Abelev *et al.* (STAR Collaboration), Phys. Rev. C **81**, 054908 (2010); B. I. Abelev *et al.* (STAR Collaboration), Phys. Rev. Lett. **103**, 251601 (2009).
[5] D. E. Kharzeev and H. J. Warringa, Phys. Rev. D **80**, 034028 (2009).
[6] I. Amado, K. Landsteiner, and F. Pena-Benitez, J. High Energy Phys. 05 (2011) 081.
[7] N. P. Landsman and C. G. van Weert, Phys. Rep. **145**, 141 (1987).
[8] R. Delbourgo and A. Salam, Phys. Lett. B **40**, 381 (1972); T. Eguchi and P. G. O. Freund, Phys. Rev. Lett. **37**, 1251 (1976); L. Alvarez-Gaume and E. Witten, Nucl. Phys. **B234**, 269 (1984).
[9] For a discussion of the gravitational anomaly, see, e.g., section three of the third reference in [8].
[10] K. Landsteiner, E. Megias, L. Melgar, and F. Pena-Benitez, arXiv:1107.0368.
[11] Y. Neiman and Y. Oz, J. High Energy Phys. 03 (2011) 023.
[12] D. T. Son and A. R. Zhitnitsky, Phys. Rev. D **70**, 074018 (2004).
[13] B. Keren-Zur and Y. Oz, J. High Energy Phys. 06 (2010) 006.
[14] D. E. Kharzeev and D. T. Son, Phys. Rev. Lett. **106**, 062301 (2011).
[15] S. H. S. Alexander, M. E. Peskin, and M. M. Sheikh-Jabbari, Phys. Rev. Lett. **96**, 081301 (2006).