Device-Independent Witnesses of Genuine Multipartite Entanglement

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We consider the problem of determining whether genuine multipartite entanglement was produced in an experiment, without relying on a characterization of the systems observed or of the measurements performed. We present an n-partite inequality that is satisfied by all correlations produced by measurements on biseparable quantum states, but which can be violated by n-partite entangled states, such as Greenberger-Horne-Zeilinger states. In contrast to traditional entanglement witnesses, the violation of this inequality implies that the state is not biseparable independently of the Hilbert space dimension and of the measured operators. Violation of this inequality does not imply, however, genuine multipartite nonlocality. We show more generically how the problem of identifying genuine tripartite entanglement in a device-independent way can be addressed through semidefinite programming.

The generation of multipartite entanglement is a central objective in experimental quantum physics. For instance, entangled states of 14 ions and 6 photons have recently been produced [1,2]. In any such experiment, a typical question arises: How can we be sure that genuine n-partite entanglement was present? An n-partite state is said to be (genuinely) n-partite entangled if it is not biseparable, that is, if it cannot be prepared by mixing states that are separable with respect to some partition. For instance, a tripartite state ρ_{bs} is biseparable if it admits a decomposition

$$\rho_{\mathrm{bs}} = \sum_k \rho_{AB}^k \otimes \rho_C^k + \sum_k \rho_{AC}^k \otimes \rho_B^k + \sum_k \rho_{BC}^k \otimes \rho_A^k, \quad (1)$$

where the weight of each state in the mixture has been included in its normalization; a state that cannot be written as above is tripartite entangled. Determining whether n-partite entanglement was produced in an experiment represents a difficult problem that has drawn much attention lately (see, e.g., [3,4]). The usual approach consists of measuring a witness of genuine multipartite entanglement, or of doing a full state tomography followed by a direct analysis of the reconstructed density matrix.

Such approaches, however, not only rely on the observed statistics to deduce the presence of entanglement, but also require a detailed characterization of the systems observed and of the measurements performed. Consider, e.g., the following witness of genuine tripartite entanglement:

$$M = X_1 X_2 X_3 - X_1 Y_2 Y_3 - Y_1 X_2 Y_3 - Y_1 Y_2 X_3, \quad (2)$$

where $X_j = \sigma_x$ and $Y_j = \sigma_y$ are the Pauli spin observables in the x and y direction for particle j. For any biseparable three-qubit state $\langle M \rangle = \mathrm{tr}(M\rho) \leq 2$ [5]. Thus if measurements on three spin-$\frac{1}{2}$ particles give an average value $\langle M \rangle > 2$, we can conclude that the state is tripartite entangled.

Suppose, however, that the measurement Y_3 carries a slight (possibly unnoticed) bias towards the x direction, i.e., we actually measure $Y_3 = \cos\theta\sigma_y + \sin\theta\sigma_x$. Then one sees that, all other measurements being ideal, the biseparable state $|\psi\rangle = \frac{1}{2}(|00\rangle + e^{-i\phi}|11\rangle)_{AB} \otimes (|0\rangle + |1\rangle)_C$, where $\phi = \arctan(\sin\theta)$, yields $\langle M \rangle = 2\sqrt{1 + \sin^2\theta}$, which is strictly larger than 2 for any $\theta \neq 0$. Thus, unless we measure all particles exactly along the x and y directions, we can no longer conclude that observing $\langle M \rangle > 2$ implies tripartite entanglement. Importantly, this is not a unique feature of the above witness, but rather all conventional witnesses are, to some extent, susceptible to such systematic errors that are seldom taken into account.

Furthermore, tomography and usual entanglement witnesses typically assume that the dimension of the Hilbert space is known. For instance, in an experiment demonstrating, say, entanglement between four ions, we usually view each ion as a two-level system. But an ion has many degrees of freedom (position, vibrational modes, internal energy levels, etc.). Given the inevitable imperfections in experiment, is it justifiable to treat the relevant Hilbert space of each ion as two dimensional and how does this simplification affect our conclusions about the entanglement present in the system [6]? Even if it is justified to view each ion as a qubit, is entanglement between four systems really necessary to reproduce the measurement data, or could they be reproduced with fewer entangled systems if qutrits were manipulated instead?

These remarks motivate the introduction of entanglement witnesses that are able to guarantee n-partite entanglement, without relying on the types of measurements performed, the precision involved in their implementation, or on assumptions about the relevant Hilbert space dimension. We call such witnesses device-independent entanglement witnesses (DIEW). This type of approach was already considered in [7,8]. Note that other solutions to

the above problems are possible, such as entanglement witnesses tolerating a certain misalignment in the measurement apparatuses [9] or the characterization of realistic measurement apparatuses through squashing maps [6]. These other approaches, however, still require some partial characterization of the system and measurement apparatuses, which is not necessary when using DIEWs.

Any DIEW is a Bell inequality (i.e., a witness of nonlocality). Indeed, (i) the violation of a Bell inequality implies the presence of entanglement, and (ii) any measurement data that do not violate any Bell inequality can be reproduced using quantum states that are fully separable [10]. The violation of a Bell inequality is thus a necessary and sufficient condition for the detection of entanglement in a device-independent (DI) setting. This observation is the main insight behind DI quantum cryptography [11,12], where the presence of entanglement is the basis of security.

The relation between DIEW for n-partite entanglement and witnesses of multipartite nonlocality is more subtle. While there exist Bell inequalities that detect genuine n-partite nonlocality [13–15], not every DIEW for n-partite entanglement corresponds to such a Bell inequality. Consider, for instance, the expression (2). If no assumptions are made on the type of systems observed and measurements performed, the inequality $\langle M \rangle \leq 2$ corresponds to Mermin's Bell-type inequality [16]; i.e., a value $\langle M \rangle > 2$ necessarily reveals nonlocality, hence entanglement. Moreover, a value $\langle M \rangle > 2\sqrt{2}$ guarantees tripartite entanglement [7,14]. The Mermin expression (2) can thus be used as a tripartite DIEW. Yet, it cannot be used as a Bell inequality for genuine tripartite nonlocality, since a simple model involving communication between two parties only already achieves the algebraic maximum $\langle M \rangle = 4$ [14].

The objectives of this Letter are to formalize the concept of DIEW for genuine multipartite entanglement and initiate a systematic study that goes beyond the early examples [7,8]. We will start by introducing the notion of quantum biseparable correlations. We then present a simple DIEW for n-partite entanglement which is stronger for Greenberger-Horne-Zeilinger (GHZ) states than all the inequalities introduced in [7,8]. For $n = 3$, we also provide a general method for determining whether given correlations reveal tripartite entanglement and apply it to GHZ and W states. Apart from yielding practical criteria for the characterization of entanglement in a multipartite setting, our results also clarify the relation between device-independent multipartite entanglement and mulitpartite nonlocality.

Biseparable quantum correlations.—For simplicity of exposition, let us consider an arbitrary tripartite system (the following discussion easily generalizes to the n-party case). To characterize in a DI way its entanglement properties, we consider a Bell-type experiment: on each subsystem, one of m possible measurements is performed, yielding one of d possible outcomes. We label the measurements on each of the three subsystems by

$x, y, z \in \{1, \ldots, m\}$ (corresponding, e.g., to the values of macroscopic knobs on the measurement apparatuses) and denote the corresponding classical outcomes $a, b, c \in \{1, \ldots, d\}$. The correlations obtained in the experiment are characterized by the joint probabilities $P(abc|xyz)$ of finding the triple of outcomes a, b, c given the measurement settings x, y, z.

We say that $P(abc|xyz)$ are biseparable quantum correlations if they can be reproduced through local measurements on a biseparable state ρ_{bs}, i.e., if there exists a biseparable quantum state (1) in some Hilbert space $\mathcal{H}$, measurement operators $M_{a|x}$, $M_{b|y}$, and $M_{c|z}$ (which without loss of generality we can take to be projections satisfying $M_{a|x}M_{a'|x} = \delta_{a,a'}M_{a|x}$ and $\sum_a M_{a|x} = \mathbb{1}$), such that

$$P(abc|xyz) = \text{tr}[M_{a|x} \otimes M_{b|y} \otimes M_{c|z}\rho_{bs}]. \qquad (3)$$

If given quantum correlations $P(abc|xyz)$ are not biseparable, they necessarily arise from measurements on a tripartite-entangled state, and this conclusion is independent of any assumptions on the type of measurements performed or on the Hilbert space dimension.

Equivalently, biseparable quantum correlations can be defined as those that can be written in the form

$$P(abc|xyz) = \sum_k P_Q^k(ab|xy)P_Q^k(c|z)$$
$$+ \sum_k P_Q^k(ac|xz)P_Q^k(b|y)$$
$$+ \sum_k P_Q^k(bc|yz)P_Q^k(a|x), \qquad (4)$$

where $P_Q^k(ab|xy)$ and $P_Q^k(c|z)$ correspond, respectively, to arbitrary two-party and one-party quantum correlations; i.e., they are of the form $P_Q^k(ab|xy) = \text{tr}[M_{a|x}^k \otimes M_{b|y}^k \rho_{AB}^k]$ and $P_Q^k(c|z) = \text{tr}[M_{c|z}^k \rho_C^k]$ for some unnormalized quantum states ρ_{AB}^k, ρ_C^k and measurement operators $M_{a|x}^k$, $M_{b|y}^k$, $M_{c|z}^k$ [and similarly for the other terms in (4)]. Here, the measurement operators for different bipartitions need not be the same (though this can always be achieved as shown in Sec. D of [17]). Clearly, from the definition (1) of biseparable states, any correlations of the form (3) are of the form (4). Conversely, any correlations of the form (4) are also of the form (3), see Sec. A of [17].

Let Q_3 denote the set of tripartite quantum correlations and $Q_{2/1} \subset Q_3$ the set of biseparable quantum correlations. From (4), it is clear that $Q_{2/1}$ is convex and that its extremal points are of the form $P_Q^{ext}(ab|xy)P_Q^{ext}(c|z)$, where $P_Q^{ext}(ab|xy)$ is an extremal point of the set Q_2 of bipartite quantum correlations and $P_Q^{ext}(c|z)$ an extremal point of the set Q_1 of single-party correlations (the extreme points of Q_1 are actually classical, deterministic points). Since the set $Q_{2/1}$ is convex, it can be characterized by linear inequalities. Those linear inequalities separating $Q_{2/1}$ from Q_3 correspond to DIEWs for tripartite entanglement. Since $Q_{2/1}$ has an infinite number of extremal points, there exist

an infinite number of such inequalities. Note that the set of local correlations $P(abc|xyz) = \sum_k P^k(a|x)P^k(b|y)P^k(c|z)$ is contained in $Q_{2/1}$. Hence, any DIEW for tripartite entanglement is a Bell inequality (though not necessarily a tight one). Note also that the decomposition (4) corresponds to a Svetlichny-type decomposition [13] where all bipartite factors are restricted to be quantum, whereas less restrictive constraints [or even none in Svetlichny's original definition $P(abc|xyz) = \sum_k P^k(ab|xy)P^k(c|z) + \cdots$] are imposed on these bipartite terms in the definitions of multipartite nonlocality [13]. It follows that the set of genuinely tripartite nonlocal correlations is larger than the set of biseparable quantum correlations as illustrated in Fig. 1. Thus, while any Bell inequality detecting genuine tripartite nonlocality is a DIEW for tripartite entanglement, the converse is not necessarily true. All these observations extend to the n-party case.

A DIEW for n-partite entanglement.—We now present a DIEW for n parties, where each party i performs a measurement $x_i \in \{1, 2, 3\}$ and obtains an outcome $a_i \in \{-1, 1\}$. We denote $E(\bar{x})$ the correlator associated to the measurement settings $\bar{x} = (x_1 \ldots, x_n)$, i.e., the expectation value $E(\bar{x}) = \sum_{\bar{a}} P(\bar{a}|\bar{x}) \prod_{i=1}^n a_i$, where $\bar{a} = (a_1 \ldots, a_n)$ denotes an n-tuple of outcomes. Let $E_n^k = \sum_{\bar{x}} \delta(\sum_{i=1}^n x_i = k)E(\bar{x})$ be the sum of correlators $E(\bar{x})$ for which the measurement settings x_i of the n parties sum up to k. Let f_k be a function such that $f_{k+3} = -f_k$ and take successively the values $[1, 1, 0]$ on the integers $k = 0, 1, 2$. Then the inequality

$$I_n = \sum_{k=n}^{3n} f_{k-n} E_n^k \leq 2 \times 3^{n-3/2} \qquad (5)$$

is satisfied by all biseparable quantum correlations, and is thus a DIEW for genuine n-partite entanglement. The
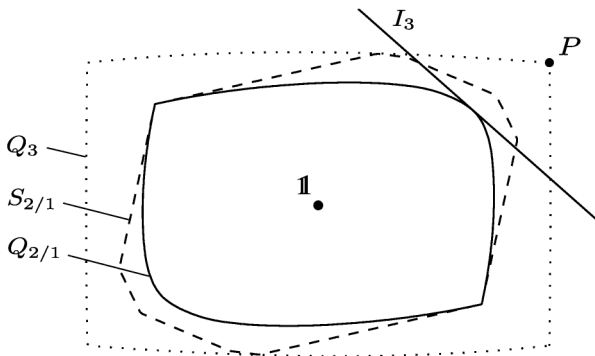

FIG. 1. A particular slice of the space of tripartite correlations with three settings and two outcomes representing schematically the sets of general quantum (Q_3), Svetlichny ($S_{2/1}$), and biseparable quantum correlations ($Q_{2/1}$). The point $\mathbb{1}$ corresponds to random correlations and P to the GHZ correlations maximally violating the DIEW (5), which is represented by the straight line I_3; note that a DIEW can be violated by Svetlichny-local correlations. (The Svetlichny polytope $S_{2/1}$ can be determined exactly using linear programming, while Q_3 and $Q_{2/1}$ can be approximated efficiently using SDP techniques; see main text.)

proof of this statement is based on the decomposition (4) and is given in Sec. B of [17]. The Svetlichny bound associated to the expression I_n, on the other hand, is easily found to be $4 \times 3^{n-2} > 2 \times 3^{n-3/2}$ (see Sec. B of [17]); for the local bound of I_n, see [18].

We now illustrate how this DIEW can be used to detect genuine multipartite entanglement. For this, let us consider the noisy GHZ state $\rho = V|\text{GHZ}\rangle_{nn}\langle\text{GHZ}| + (1 - V)\mathbb{1}/2^n$ characterized by the visibility V. Carrying out the measurements $\cos[(\frac{x_i-1}{3} - \frac{1}{6n})\pi]\sigma_x + \sin[(\frac{x_i-1}{3} - \frac{1}{6n})\pi]\sigma_y$ on all parties, we obtain $I_n = 3^{n-1/2}V$, which violates (5) provided that $V > 2/3$. The DIEW (5) can thus detect in a DI way n-partite entanglement in a noisy GHZ state for visibilities as low as $V = 2/3$. This significantly improves over the threshold visibility $V = 1/\sqrt{2}$ required to violate the DIEW based on the Mermin expression (2) or the different inequalities introduced in [8]. Note that in a DI setting it is not possible to detect the tripartite entanglement in tripartite GHZ states below $V = 1/2$ using projective measurements; in this case, there exists a biseparable model reproducing all GHZ correlations (see Sec. C of [17]).

In the case $n = 3$, the DIEW (5) takes the form $I_3 = E_3^3 + E_3^4 - E_3^6 - E_3^7 + E_3^9 \leq 6\sqrt{3}$. It therefore involves only 18 expectation values, compared to 27 for a full tomography of a three-qubit system. Let us stress, however, that contrary to usual entanglement witnesses I_3 is not restricted to two-dimensional Hilbert spaces, even though it uses observables with binary outcomes. For instance, if all parties perform the measurements $2|\phi(x_i)\rangle\langle\phi(x_i)| - \mathbb{1}$ with $|\phi(x_i)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i(6x_i-7)\pi/18}|1\rangle)$ on the three-qutrit state $\frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$, then $I_3 = 6\sqrt{3} + 8/3$, showing that the state is tripartite entangled.

General characterization of biseparable quantum correlations in the case n = 3.—Though the DIEW (5) seems particularly well adapted to GHZ states, we cannot expect a single nor a finite set of DIEW to completely characterize the biseparable region, as illustrated in Fig. 1. It is thus desirable to derive a general method to decide whether arbitrary correlations are biseparable. Here we show how the semidefinite programming (SDP) techniques introduced in [19,20] can be used to certify that the correlations observed in an experiment are tripartite entangled.

Our approach is based on the observation that the tensor product separation $\rho_{AB} \otimes \rho_C$ at the level of states, cf. definition (1), can be replaced by a commutation relation at the level of operators. Specifically, let $s = \{AB/C, AC/B, BC/A\}$ denote the three possible partitions of the parties into two groups. Then, $P(abc|xyz)$ are biseparable quantum correlations if and only if there exist three arbitrary (not necessarily biseparable) states ρ^s and three sets of measurement operators $\{M_{a|x}^s, M_{b|y}^s, M_{c|z}^s\}$ such that

$$P(abc|xyz) = \sum_s \text{tr}[M_{a|x}^s \otimes M_{b|y}^s \otimes M_{c|z}^s \rho^s], \qquad (6)$$

where measurement operators corresponding to an isolated party commute, i.e., $[M_{c|z}^{AB/C}, M_{c'|z'}^{AB/C}] = 0$, and similarly for the other partitions. The equivalence between (3) and (6) is established in Sec. D of [17]. The problem of determining whether given correlations $P(abc|xyz)$ are biseparable thus amounts to finding a set of operators satisfying a finite number of algebraic relations (the projection defining relations of the type $M_{a|x}^s M_{a'|x}^s = \delta_{a,a'} M_{a|x}^s$, $\sum_a M_{a|x}^s = \mathbb{1}$ and the commutation relations mentioned above) such that (6) holds. Such a problem is a typical instance of the SDP approach introduced in [19,20] (see Sec. D of [17]). Specifically, it follows from the results of [20] that one can define an infinite hierarchy of criteria that are necessarily satisfied by any correlations of the form (6) and which can be tested using SDP. If given correlations do not satisfy one of these criteria, we can conclude that they reveal tripartite entanglement. Further, it is possible in this case to derive an associated DIEW from the solution of the dual SDP. Modulo a technical assumption, it can be shown that the hierarchy of SDP criteria is complete; that is, if given correlations are not biseparable this will necessarily show up at some finite step in the hierarchy.

Application to GHZ and W states.—Using finite levels of this hierarchy and optimizing over the possible measurements, we investigated the minimal visibilities above which the GHZ state $|GHZ\rangle$ and the W state $|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$ exhibit correlations that are not biseparable (and thus reveal genuine tripartite entanglement) in the case of two and three measurement settings per party. Our results are summarized in Table I. For GHZ states, the reported visibility $V_{\min} = 2/3$ for three measurements per party corresponds to the threshold required to violate the DIEW (5), suggesting that this DIEW is optimal in this case. In the case of two measurements per party, we could not lower the visibility below the threshold $V_{\min} = 1/\sqrt{2}$, which corresponds to the visibility required to violate the DIEW based on Mermin expression (2) and the DIEWs introduced in [7,8]. Note, however, that for $V > 1/\sqrt{2}$ the GHZ state violates Svetlichny's inequality [13] and thus exhibits genuine tripartite nonlocality. Thus for the GHZ state the DIEWs introduced in [7,8] do not improve over what can already be concluded using the standard notion of tripartite nonlocality. On the other hand, our numerical explorations suggest that the visibilities $V_{\min} = 2/3$ for GHZ states with three measurements and $V_{\min} = 3/4$ for W states with two measurements cannot be attained using the notion of genuine tripartite nonlocality, illustrating the interest of the weaker notion of DIEW.

Discussion.—To conclude, we comment on some possible directions for future research. First of all, note that by identifying the measurement settings "X_i" with $x_i = 1$ and "Y_i" with $x_i = 2$, the two-setting DIEW based on Mermin expression (2) can be written as $E_3^3 - E_3^5 \leq 2\sqrt{2}$, which is of the same general form as the three-setting DIEW (5).

TABLE I. Summary of numerical investigations.

State	$V_{\min}$ with two settings	$V_{\min}$ with three settings
GHZ	$0.7071 \simeq 1/\sqrt{2}$	$0.6667 \simeq 2/3$
W	$0.7500 \simeq 3/4$	0.7158

This suggests that the DIEWs based on (2) and (5) actually form part of a larger family of m-settings DIEWs. This deserves further investigation. A second problem is to derive simple DIEWs that are adapted to W states and that, in particular, reproduce the threshold visibilities obtained in Table I. Finally, we have shown a practical method to characterize tripartite biseparable correlations using SDP. It would be interesting to understand how this generalizes to the n-partite case. A possibility would be to combine the approach of [19,20] with the symmetric extensions of [21]. This question will be investigated elsewhere.

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