Brownian Motion with Active Fluctuations

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We study the effect of different types of fluctuation on the motion of self-propelled particles in two spatial dimensions. We distinguish between passive and active fluctuations. Passive fluctuations (e.g., thermal fluctuations) are independent of the orientation of the particle. In contrast, active ones point parallel or perpendicular to the time dependent orientation of the particle. We derive analytical expressions for the speed and velocity probability density for a generic model of active Brownian particles, which yields an increased probability of low speeds in the presence of active fluctuations in comparison to the case of purely passive fluctuations. As a consequence, we predict sharply peaked Cartesian velocity probability densities at the origin. Finally, we show that such a behavior may also occur in non-Gaussian active fluctuations and discuss briefly correlations of the fluctuating stochastic forces.

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In recent decades there has been an increasing focus on statistical descriptions of systems far from equilibrium. A whole class of biological and physical systems which may be referred to as active matter has been studied theoretically and experimentally. Examples of such systems range from the dynamical behavior of individual units such as Brownian motors [1,2], motile cells [3–8], macroscopic animals [9–11], or artificial self-propelled particles [12–16] to large coupled ensembles of such units and their large scale collective dynamics [11,17–21]. A major driving force of the active matter research is continuously improving experimental techniques, such as, for example, automated digital tracking [6,8,10,19] or the realization of active granular and colloidal systems [12,13,16,22].

Despite recent advances in active matter research, there is still a lack of theoretical foundations. For example, up to date no clear distinction has been made between external fluctuations due to a fluctuating environment and internal fluctuations originating from the active nature of the system. Only recently it was shown how internal fluctuations may lead to a complex behavior of the mean squared displacement of active particles [23].

Recent measurements of stationary speed probability densities of active biological agents report increased probabilities of low speeds $|\mathbf{v}| \simeq 0$, which cannot be explained by the Rayleigh probability density in 2D or the Maxwell probability density in 3D [6–8]. Motivated by these results, we focus in this Letter on the impact of active (out-of-equilibrium) fluctuations on velocity and speed probability density functions (PDFs). We will show in a generic model that active fluctuations have a characteristic impact on the stationary speed PDF accessible in experiments.

We assume equations of motion for a fluctuating particle of the following form:

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad m\dot{\mathbf{v}} = -\gamma(\mathbf{v})\mathbf{v} + \boldsymbol{\eta}(t), \tag{1}$$

with $\mathbf{r}(t) = (x(t), y(t))$ being the position vector of the particle, and $\mathbf{v}(t) = (v_x(t), v_y(t))$ being its velocity vector. We set the particle mass without loss of generality to $m = 1$. The first term in the velocity equation models the friction or propulsion force, which acts along the direction of the velocity vector. For active particles, the coefficient $\gamma(\mathbf{v})$ depends in general on the velocity of the particle. It can even change sign and if it becomes negative, the velocity is amplified due to a nonequilibrium propulsion mechanism. The second term is a random force $\boldsymbol{\eta}(t)$ accounting for the stochasticity in the motion of individual particles.

Here we consider polar active particles with a preferred direction of motion (heading) determined by their propulsion mechanism (“head-tail” asymmetry). The orientation of the particle is defined by the unit (heading) vector $\mathbf{e}_h$. In two spatial dimensions, it is fully determined by the angle $\varphi(t)$ defining the polar orientation with respect to the x axis: $\mathbf{e}_h(t) = (\cos\varphi(t), \sin\varphi(t))$, and the evolution of the position of can be rewritten as $\dot{\mathbf{r}}(t) = \mathbf{v}(t) = v(t)\,\mathbf{e}_h(t)$, with $v(t)$ being the velocity with respect to the heading.

The temporal evolution of the velocity vector in the new variables $(v(t), \varphi(t))$ reads

$$\dot{v} = -\gamma(v)v + \mathbf{e}_h \cdot \boldsymbol{\eta}(t), \qquad \dot{\varphi} = \frac{1}{v}\mathbf{e}_\varphi \cdot \boldsymbol{\eta}(t), \tag{2}$$

where $\mathbf{e}_\varphi(t) = (-\sin\varphi(t), \cos\varphi(t))$ is the unit vector in the angular direction [24]. We emphasize the difference of velocity with respect to the heading v and speed $s = |v| = |\mathbf{v}|$. The radial component of velocity $v = \mathbf{v}\mathbf{e}_h$ might take also negative values, corresponding to a reverse motion of the particle with respect to its heading, whereas the speed given by the absolute value of the velocity is always positive. Thus the v, φ coordinates have to be distinguished from classical polar coordinates (s, ϕ).

In this Letter, we distinguish passive and active fluctuations: (i) Passive fluctuations have their origin in a fluctuating environment in which the particle moves. In a homogeneous environment the passive random force $\boldsymbol{\eta}_p(t)$ is independent on the direction of motion (heading) of the particle. A classical example of passive fluctuations is ordinary Brownian motion, where the stochastic force is associated with random collisions of fluid molecules with the Brownian particle. The fluctuating force reads

$$\boldsymbol{\eta}_p(t) = \sqrt{2D}\boldsymbol{\xi}(t) = \sqrt{2D}[\xi_x(t)\mathbf{e}_x + \xi_y(t)\mathbf{e}_y], \quad (3)$$

where D is the noise intensity and ξ_i are normally distributed random variables with $\langle\xi_i(t)\rangle = 0$ and $\langle\xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$.

In contrast to Eq. (3), we introduce (ii) active fluctuations as independent stochastic processes in the direction of motion and in the velocity with respect to the heading [23] and perpendicular to it. A simple realization is independent Gaussian white noise in the direction of motion $\mathbf{e}_h$ and in the angular direction $\mathbf{e}_\varphi$:

$$\boldsymbol{\eta}_a(t) = \sqrt{2D_v}\xi_v(t)\mathbf{e}_h + \sqrt{2D_\varphi}\xi_\varphi(t)\mathbf{e}_\varphi \quad (4)$$

with angular and velocity noise intensities: D_φ, D_v. Hence, active fluctuations are a pure out-of-equilibrium phenomenon. They are relevant in the motion of biological agents or self-propelled particles. The origin of these fluctuations can be, for example, variations in the propulsion of chemically powered colloids [12–14], complex intracellular processes in cell motility [6,8], or unresolved internal decision processes in animals [10,11].

We point out that despite the similarity of Eqs. (3) and (4) the active fluctuations lead to multiplicative noise terms in Cartesian coordinates [Eq. (1)]. The basis $(\mathbf{e}_h, \mathbf{e}_\varphi)$ depends on the actual orientation $\varphi(t)$ in contrast to $(\mathbf{e}_x, \mathbf{e}_y)$ which is fixed. However, in the $v\varphi$ frame the equations of motion for active fluctuation assume a simple form with additive noise terms:

$$\dot{v} = -\gamma(v)v + \sqrt{2D_v}\xi_v(t), \qquad \dot{\varphi} = \frac{1}{v}\sqrt{2D_\varphi}\xi_\varphi(t). \quad (5)$$

The v dynamics decouple from the φ dynamics. We can easily derive the stationary PDF of velocities via the corresponding Fokker-Planck equation. For the spatially isotropic case there is no preferred direction of motion and over time particles move randomly in all directions.

In general, the motion of a small particle will be influenced by both fluctuations types $\boldsymbol{\eta}(t) = \boldsymbol{\eta}_a(t) + \boldsymbol{\eta}_p(t)$. The passive fluctuations introduce multiplicative noise terms in $v\varphi$ frame (2). Using Stratonovich interpretation the corresponding Fokker-Planck equation can be derived to

$$\frac{\partial\hat{p}(v,\varphi,t)}{\partial t} = -\frac{\partial}{\partial v}\left[\left(-\gamma(v)v + \frac{D}{v}\right)\hat{p}\right] + (D_v + D)\frac{\partial^2\hat{p}}{\partial v^2} + \frac{D_\varphi + D}{v^2}\frac{\partial^2\hat{p}}{\partial\varphi^2}. \quad (6)$$

The resulting stationary PDF in φ is uniform $q(\varphi) = 1/(2\pi)$ and the total stationary PDF decomposes as $\hat{p}(v,\varphi) = p(v)q(\varphi)$ with

$$p(v) \sim v^{D/D+D_v}\exp\left(-\int^v dv'\frac{\gamma(v')v'}{(D_v + D)}\right). \quad (7)$$

The speed PDF $\tilde{p}(s)$ can be derived from $p(v)$ by taking into account that on average for each polar angle ϕ some particles head in the direction of $\varphi = \phi$, whereas the others have the opposite heading $\varphi = \phi + \pi$. In the presence of both fluctuation types (mixed case) and for arbitrary $\gamma(v)$, the stationary speed PDF can be calculated using Eq. (7): $\tilde{p}(s) = p(s) + p(-s)$. In particular, we consider here a linear velocity dependent friction [25,26]: $\gamma(v) = \gamma_0(1 - v_0/v)$, with $v_0 \geq 0$ being the stationary velocity of the particle with respect to its heading. The constant γ_0 is the inverse relaxation time of the velocity dynamics. The dissipative force $-\gamma(v)v = \gamma_0(v_0 - v)$ is negative for $v > v_0$ (friction) and positive for $v < v_0$ (propulsion). It reduces to Stokes friction for $v_0 = 0$. The stationary speed PDF reads

$$\tilde{p}(s) \sim s^{D/D+D_v}[e^{-A(s-v_0)^2} + e^{-A(s+v_0)^2}], \quad (8)$$

with $A = \gamma_0/2(D + D_v)$. For vanishing fluctuation strength ($D, D_v \to 0$) the speed PDF converges to δ peak at $s = v_0$ irrespective on the type of fluctuations thus the different impact of active and passive fluctuations becomes apparent only for sufficiently large fluctuation strengths, which should be expected in corresponding systems (see, e.g., [6,8,14]).

The speed PDF in Eq. (8) increases for low speeds s^α with $\alpha = D/(D + D_v)$ $(0 \leq \alpha \leq 1)$. The extreme α values correspond to the limiting cases of pure active fluctuations $\alpha = 0$ and pure passive fluctuations $\alpha = 1$. The PDF assumes a finite value at vanishing speed ($\tilde{p}(0) > 0$) only for pure active fluctuations $D = 0$, but we observe for $D > 0$ an increasing probability (integral of the PDF over a finite interval) of low speeds with increasing strength of active velocity fluctuations D_v (Fig. 1).

In the limit $D_v, D \gg \gamma v_0^2$ the self-propulsion becomes negligible and the PDF in Eq. (8) converges towards the Stokes limit ($v_0 = 0$) under the influence of both fluctuation types (see Figs. 1 and 2). The Stokes limit of pure active or passive fluctuations demonstrates clearly the different impact of the two fluctuations types. The speed PDF for purely passive fluctuations ($D_v = 0$) and $\gamma(v) = \gamma_0 = \text{const} > 0$ is given by the Rayleigh PDF $\tilde{p}_p(s) \sim s\exp[-\gamma_0 s^2/(2D)]$. The speed PDF vanishes for $s = 0$ due to the random "kicks" which are independent of the heading of the particle and drive the particle speed away

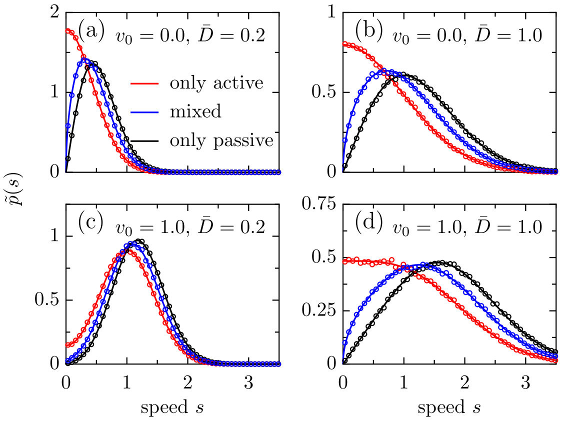

FIG. 1 (color online). Stationary speed probability density $\tilde{p}(s)$ for Stokes friction $v_0 = 0$ (a),(b) and self-propelled Brownian motion $v_0 = 1.0$ (c),(d) under the influence of active and passive fluctuations for different velocity fluctuation strengths $\bar{D} = (D + D_v)/\gamma_0$. Three cases are shown: only active fluctuation with $D = 0$, $\bar{D} = D_v/\gamma_0$ (red or light gray), only passive fluctuations $\bar{D} = D/\gamma_0$, $D_v = 0$ (black) and the mixed case $D_v = D$ (blue or dark gray); (a),(c) $\bar{D} = 0.2$; (b),(d) $\bar{D} = 1.0$. The solid lines show analytical results [Eq. (8)] whereas symbols show results obtained from Langevin simulations.

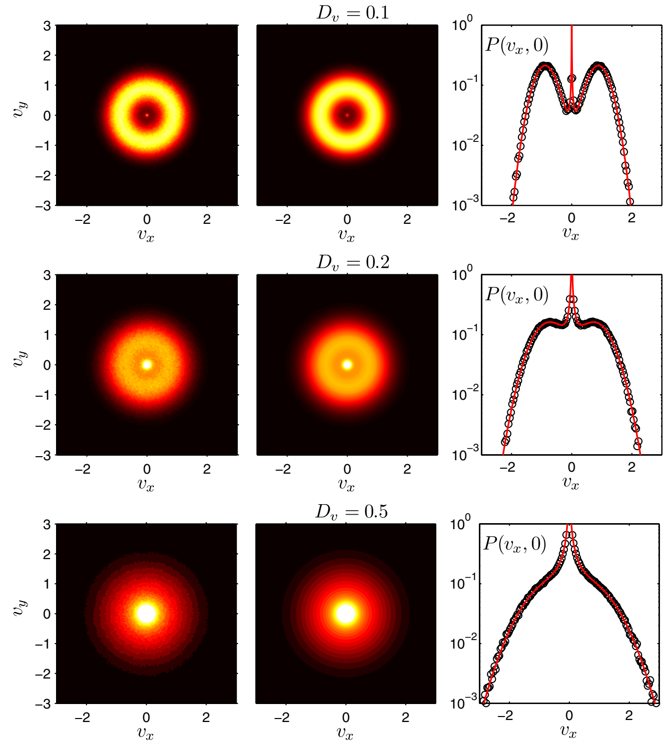

FIG. 2 (color online). Stationary Cartesian probability density $P(v_x, v_y)$ for self-propelled particles with $v_0 = 1$ and pure active fluctuations ($D = 0$) for different velocity noise strengths: $D_v = 0.1$, top; $D_v = 0.2$, center; $D_v = 0.5$ bottom. Left column: Results of numerical integration of Eqs. (1); Central column: Analytical solution given in Eq. (9); Right column: One-dimensional cross sections $P(v_x, 0)$ with analytical solutions (solid lines) and numerical results (symbols).

from $s = 0$ (Fig. 1). In the opposite case of pure active fluctuations ($D = 0$) the speed PDF is given by a half-Gaussian $\tilde{p}(s) \sim \exp[-\gamma_0 s^2/(2D_v)]$. Pure active velocity fluctuations acting along the heading direction result in a finite speed PDF at $s = 0$.

In many experiments the direct measurement of the particle heading can be very difficult. For example, the direction of motion of amoeboid cells is determined by the polarity of their intracellular cytoskeleton invisible during optical recording of cell trajectories. In this case, the Cartesian velocity PDF is a reliable measure [8]. For the mixed case with self-propulsion, it reads

$$P(v_x, v_y) \sim |\mathbf{v}|^{-D_v/D+D_v}[e^{-A(|\mathbf{v}|-v_0)^2} + e^{-A(|\mathbf{v}|+v_0)^2}], \quad (9)$$

with $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$. It can be immediately seen that a singularity of $P(v_x, v_y)$ at $|\mathbf{v}| = 0$ exists in the presence of active velocity fluctuations $D_v > 0$ (see Fig. 2) leading to a sharply peaked PDF close to the origin. Please note that this singularity follows directly from the increased probability of low speeds resulting from Eq. (8). It is a general feature of active Gaussian velocity fluctuations acting along the heading direction and is independent on the particular choice of the friction function $\gamma(v)$.

The consideration of active fluctuations, which are uncorrelated in the heading and angular direction, may be too simplistic. Therefore, we have analyzed also the situation of correlated Gaussian fluctuations. For the simplest case of constant correlations with $\langle \xi_v(t)\xi_\varphi(t')\rangle = D_{v\varphi}\delta(t - t')$, we obtain an additional fluctuation induced torque leading to a finite mean turning velocity $\langle \dot{\varphi}\rangle = -D_{v\varphi}/(2v^2)$. Since such a preferred turning was not reported so far, we conclude that correlations will have a more complex temporal behavior with vanishing mean. Thus the assumption of independent fluctuations is reasonable as confirmed by experiments [6].

As an example of a system with active non-Gaussian velocity fluctuations, we consider a particle driven by shot noise $\xi_{SN}(t)$ consisting of short force impulses at t_i with random exponentially distributed times between successive kicks [27,28]. The temporal density of the random impulses is λ. The angular dynamics is determined again by Gaussian active fluctuations. The equations of motion read

$$\dot{v} = -\gamma_0 v + D_s \xi_{SN}(t), \qquad \dot{\varphi} = \frac{1}{v}\sqrt{2D_\varphi}\xi_\varphi(t), \quad (10)$$

with $\xi_{SN}(t) = \sum_i \sigma_i \delta(t - t_i)$ where the σ_i are exponentially distributed amplitudes. We set $\langle \sigma \rangle = 1$, therefore $D_s \geq 0$ denotes the strength of the kicks. The shot-noise process has a nonvanishing mean $\langle \xi_{SN}\rangle = \lambda$. The random force is always positive, $D_s \xi_{SN}(t) \geq 0$, therefore the velocity assumes only positive values yielding $v = s$ at all times. The stationary speed PDF can be obtained if the shot noise is taken as a limit of a Markovian telegraph process [27,28]. In the corresponding white shot-noise limit the PDF becomes

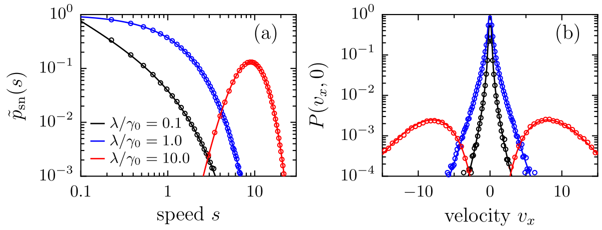

FIG. 3 (color online). (a) Stationary speed PDF $\tilde{p}_s(s)$ for active shot-noise fluctuations. Solid lines show the result obtained in Eq. (11), whereas symbols show numerical results. (b) The corresponding one-dimensional cross section of Cartesian velocity distributions.

$$\tilde{p}_{\rm SN}(s) = N_{\rm SN} s^{(\lambda/\gamma_0)-1} \exp\left[-\frac{s}{D_s}\right], \qquad (11)$$

with $N_{\rm SN}^{-1} = D_s^{(\lambda/\gamma_0)}\Gamma(\lambda/\gamma_0)$. We distinguish two qualitatively different regimes: for $\lambda/\gamma_0 < 1$ the speed PDF assumes a maximum at $s = 0$ and decays monotonously with increasing s. For $\lambda/\gamma_0 > 1$ $p_{\rm SN}(s)$ vanishes for $s = 0$, increases initially with s up to a maximum at $s = D_s(\lambda/\gamma_0 - 1)$ and decreases for larger speeds [Fig. 3(a)]. The speed PDF increases at small speeds as s^α with $\alpha = \lambda/\gamma_0 - 1$, thus for $\lambda/\gamma_0 < 2$ the corresponding Cartesian distributions shows a divergence at the origin [Fig. 3(b)] as for active Gaussian fluctuations.

In conclusion, we distinguish active fluctuations in self-propelled Brownian motion from passive ones by formulating corresponding source terms in the Langevin equation. We analyze their impact on the stationary speed and velocity PDFs. In general, active Gaussian fluctuations acting along the direction of motion, result in a power-law increase of the speed density $\tilde{p}(s) \sim s^\alpha$ with $\alpha < 1$ for $s \ll 1$ and, as a consequence, to a strongly peaked Cartesian velocity PDF with a divergence at the origin (Fig. 2). Such peaks, corresponding to increased counts in velocity histograms at $\mathbf{v} \approx 0$, have been reported from cell experiments [7,8].

We expect that a corresponding analysis of experimentally obtained speed and velocity PDFs for various self-propelled systems will yield similar results, which in turn provide information about the fluctuations governing the microscopic dynamics. In contrast to the mean squared displacement [23], these characteristic deviations in the PDF become apparent irrespective of the time-scale separation between velocity and angular dynamics and are therefore a reliable indicator of active fluctuations. In fact, the exponent $\alpha < 1$ can be used as a quantitative measure of the relative strength of active fluctuations, if the corresponding power law at low speed can be determined reliably. Furthermore, we are convinced that a similar distinction of fluctuation in overdamped models will have a similar impact on corresponding PDFs.

Finally, we have shown that already a rather subtle difference in the implementation of fluctuation can have a dramatic impact on the motion statistics of individual units and should be taken into account, e.g., in modeling drift and diffusion of active particles.

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