Quantum Quench in the Transverse-Field Ising Chain

Pasquale Calabrese,[1] Fabian H. L. Essler,[2] and Maurizio Fagotti[1]
[1]Dipartimento di Fisica dell'Università di Pisa and INFN, Pisa, Italy
[2]The Rudolf Peierls Centre for Theoretical Physics, Oxford University, Oxford OX1 3NP, United Kingdom
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We consider the time evolution of observables in the transverse-field Ising chain after a sudden quench of the magnetic field. We provide exact analytical results for the asymptotic time and distance dependence of one- and two-point correlation functions of the order parameter. We employ two complementary approaches based on asymptotic evaluations of determinants and form-factor sums. We prove that the stationary value of the two-point correlation function is not thermal, but can be described by a generalized Gibbs ensemble (GGE). The approach to the stationary state can also be understood in terms of a GGE. We present a conjecture on how these results generalize to particular quenches in other integrable models.

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Recent experiments on trapped cold atomic gases [1,2] have raised intriguing fundamental questions regarding the nonequilibrium dynamics of correlated many-body quantum systems. These cold atom systems are sufficiently weakly coupled to their environments as to allow the observation of essentially unitary nonequilibrium time evolution on long time scales. The quantum Newton's cradle experiments of Kinoshita *et al.* [2] in particular have focused attention on the roles played by dimensionality and conservation laws. The observed absence of "thermalization" in quasi-one-dimensional condensates was attributed to the experimental system being approximately describable by a quantum integrable many-body theory. This in turn initiated vigorous research on clarifying the role played by quantum integrability in determining the stationary (late time) behavior of nonequilibrium evolution in correlated quantum systems [3–9]. The simplest way of driving a quantum system out of equilibrium is by means of a quantum quench: a system is prepared in the ground state of a given Hamiltonian $H(h_0)$, where h_0 is an experimentally tunable parameter such as a bulk magnetic field. At time $t = 0$ the parameter h_0 is changed suddenly to a different value h and one then considers the unitary time evolution of the system by means of the new Hamiltonian $H(h)$. Central issues that have been investigated are whether the system relaxes to a stationary state, and if it does, how to characterize its physical properties at late times. It is widely believed (see, e.g., [10] for a comprehensive summary) that the behavior of local observables (such as one and two-point correlation functions) can be described in terms of either an effective thermal (Gibbs) distribution or a generalized Gibbs ensemble (GGE) [3]. It has been argued that the latter arises for integrable models, while the former obtains for generic systems [3–6]. However, several recent studies [7–9] suggest that the behavior is more complicated and, in particular, depends on the initial state. Moreover, open questions remain even with regards to the very existence of stationary states. For example, the order parameters of certain mean-field models have recently been shown to display persistent oscillations [11]. Two recent works have raised another crucial issue in the debate on thermalization, namely, the role played by the considered observables [8,12]. More precisely, it was pointed out that the locality of the observable with respect to the elementary excitations is expected to affect the late time behavior of an observable after a quantum quench. In light of the available experimental and theoretical results further clarification of the role of integrability on the time evolution after a quantum quench calls for exact analytical results on "generic" correlation functions, i.e., those corresponding to observables nonlocal with respect to the elementary excitations. In the following we present such results for the particular case of the transverse-field Ising chain, which is a crucial paradigm for quantum critical behavior. While the model admits a representation in terms of free fermions, the order parameter is nonlocal with respect to the fermionic degrees of freedom, which renders it an ideal testing ground for thermalization ideas. Although the model has been widely analyzed in the context of quantum quenches [13–17], the nonequilbrium evolution of order parameter correlation functions is still not known analytically. In this Letter we present analytical results for the full asymptotic time and distance dependence of one- and two-point correlation functions of the order parameter in the thermodynamic limit after a quantum quench within the ferromagnetic phase. We also present partial results for quenches within the paramagnetic phase and across the critical point. Our results are obtained by two independent, novel methods. The first is based on the determinant representation of correlation functions characteristic of free-fermionic theories. The second is based on the form-factor approach [18] and is applicable more generally to integrable quenches in interacting quantum field theories [19]. This method complements existing analytical or seminumerical methods used for studying quantum quenches in integrable systems

[6,20], but has the advantage of providing analytic answers directly in the thermodynamic limit.

The model.—We consider the spin-$\frac{1}{2}$ transverse-field Ising chain (TFIC) Hamiltonian

$$H(h) = -\frac{1}{2}\sum_{l=-\infty}^{\infty}[\sigma_l^x\sigma_{l+1}^x + h\sigma_l^z], \quad (1)$$

which at zero temperature exhibits ferromagnetic ($h < 1$) and paramagnetic ($h > 1$) phases, separated by a quantum critical point. $H(h)$ can be diagonalized by a combination of Jordan-Wigner and Bogoliubov transformations (see, e.g., [13]). The dispersion of the elementary fermion excitations is $\epsilon_h(k) = \sqrt{h^2 - 2h\cos k + 1}$. The system is initially prepared in the ground state at a field h_0. The field is then instantaneously changed from h_0 to h and unitary time evolution with Hamiltonian $H(h)$ ensues. We are interested in the time evolution of the order parameter $\rho^x(t) \equiv \langle\sigma_l^x(t)\rangle$ and its two-point function $\rho^{xx}(\ell, t) \equiv \langle\sigma_l^x(t)\sigma_{l+\ell}^x(t)\rangle$. Because of translational invariance the one-point function is position independent and the two-point function depends only on the distance ℓ. An important role is played by the difference Δ_k of the Bogoliubov angles diagonalizing $H(h)$ and $H(h_0)$, respectively,

$$0 < \cos\Delta_k = \frac{hh_0 - (h + h_0)\cos k + 1}{\epsilon_h(k)\epsilon_{h_0}(k)} \leq 1. \quad (2)$$

Quenches within the ordered phase ($h, h_0 \leq 1$).—We find that at late times the order parameter relaxes to zero exponentially fast

$$\rho^x(t) \propto \exp\left[t\int_0^\pi \frac{dk}{\pi}\epsilon_h'(k)\ln(\cos\Delta_k)\right], \quad (3)$$

where $\epsilon_h'(k) = d\epsilon_h(k)/dk$. The two-point function of the order parameter exhibits exponential decay both in time and distance ($\theta(x)$ denotes the Heaviside function)

$$\rho^{xx}(\ell, t) \propto \exp\left[\ell\int_0^\pi \frac{dk}{\pi}\ln(\cos\Delta_k)\theta(2\epsilon_h'(k)t - \ell)\right] \times \exp\left[2t\int_0^\pi \frac{dk}{\pi}\epsilon_h'(k)\ln(\cos\Delta_k)\theta(\ell - 2\epsilon_h'(k)t)\right]. \quad (4)$$

In the $\ell \to \infty$ limit the first factor is equal to unity and $\rho^{xx}(\infty, t) = (\rho^x(t))^2$, confirming cluster decomposition in our nonequilibrium situation. Figure 1 shows a comparison of our asymptotic result for $\rho^{xx}(\ell, t)$ to numerical data, establishing the accuracy of the former even for relatively short separations and times. We note that (4) holds even for quenches to or from the quantum critical point and agrees with the general form put forward in [4] on the basis of semiclassical arguments.

The stationary state.—The result (4) allows us to make exact statements regarding thermalization in the model. The one-point function is trivially thermal, since it vanishes for $t \to \infty$ as was already pointed out in [4,17].

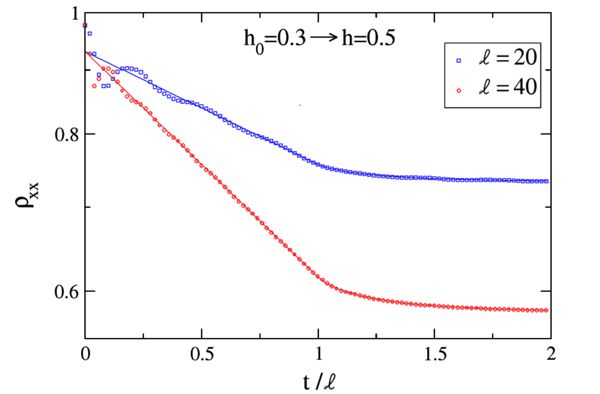

FIG. 1 (color online). $\rho^{xx}(\ell, t)$ for the quench $h_0 = 0.3 \to h = 0.5$ at fixed distance $\ell = 20$ and $\ell = 40$ against the prediction in (4). The overall amplitude of ρ^{xx} has been used as the same fit parameter in both cases.

On the other hand, in this limit the two-point function exhibits exponential decay with a correlation length

$$\xi^{-1} = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\xi^{-1}(k) = -\int_{-\pi}^{\pi}\frac{dk}{2\pi}\ln|\cos\Delta_k|. \quad (5)$$

This is reminiscent of the behavior of the equilibrium two-point function $\rho_{\rm eq}^{xx}(\ell)$ at temperature T, which decays exponentially with correlation length (for $h < 1$)[13]

$$\xi_T^{-1} = -\int_{-\pi}^{\pi}\frac{dk}{2\pi}\ln\left|\tanh\frac{\epsilon_h(k)}{2T}\right|. \quad (6)$$

For $\rho^{xx}(\ell, \infty)$ to be thermal, ξ would have to equal $\xi_{T_{\rm eff}}$, where the effective temperature $T_{\rm eff}$ is determined by the requirement that the (average) energy in the initial state $\langle\psi_0|H(h)|\psi_0\rangle$ is given by the thermal average $\langle H(h)\rangle_{T_{\rm eff}}$. This leads to the following equation fixing $T_{\rm eff}$

$$\int_{-\pi}^{\pi}\frac{dk}{2\pi}\epsilon_h(k)\cos\Delta_k = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\epsilon_h(k)\tanh\frac{\epsilon_h(k)}{2T_{\rm eff}}. \quad (7)$$

With $T_{\rm eff}$ given by (7) we find that $\xi_{T_{\rm eff}}^{-1} \neq \xi^{-1}$ and $\rho^{xx}(\ell, \infty)$ is therefore never thermal. On the other hand, for small quenches corresponding to small values of $T_{\rm eff}$, ξ and $\xi_{T_{\rm eff}}$ can be seen to coincide to order $(h - h_0)^2$ [21]. Hence there exists a small quench regime, where the thermal result provides a good approximation. This agrees with the numerical findings of Ref. [17], which were reported to be consistent with thermal behavior at low effective temperatures. In [3] it was proposed that the stationary state of integrable models can be described in terms of a GGE, defined by maximizing the entropy while keeping the energy as well as higher conservation laws fixed. For the particular case of the TFIC this results in a mode-dependent effective temperature given by

$$\cos\Delta_k = \tanh\frac{\epsilon_h(k)}{2T_{\rm eff}(k)}. \quad (8)$$

Inserting this relation into the expression (6) for ξ_T in Eq. (6) results in ξ (5), which proves that the stationary behavior after a quench is indeed described by a GGE. To the best of our knowledge this demonstrates for the first time that the GGE applies even to correlation functions of observables that are nonlocal with respect to the elementary excitations in a lattice model, thus significantly generalizing the results of [3–5]. Our expression (5) for the correlation length in terms of $T_{\mathrm{eff}}(k)$ (8) explains nicely observations made in [17], that numerical results on ρ^{xx} are described more accurately by a mode-dependent correlation length than by the thermal result (6). In [6] it was argued that the GGE quite generally describes the stationary behavior of one-point functions for quenches in integrable quantum field theories. Taking the appropriate scaling limit of our result suggests the correctness of the assumptions made in [6].

Approach to the stationary state.—Suprisingly the decay time in (3) can be similarly explained in terms of a GGE, even though it is not a property of the stationary state. The one-point function (3) is characterized by exponential decay with rate $\tau^{-1} = -\int_0^\pi \frac{dk}{\pi}\epsilon_h'(k)\ln(\cos\Delta_k)$. This can be interpreted as the average mode-dependent decay time $\tau^{-1}(k) = \epsilon_h'(k)\xi^{-1}(k)$, obtained by multiplying the mode-dependent inverse correlation length by the velocity. The relaxational behavior of the two-point function can be understood following [4] by rewriting (4) as

$$\frac{\rho^{xx}(\ell,t)}{(\rho^x(t))^2} \sim \exp\left[\int_0^\pi \frac{dk}{\pi}\left[\frac{\ell}{\xi(k)} - \frac{2t}{\tau(k)}\right]\theta(2\epsilon_h'(k)t-\ell)\right]. \tag{9}$$

The theta function expresses the fact that a given mode can only contribute to the relaxational behavior if the distance ℓ lies within its forward "light cone," while the form of the remaining factor follows from the known stationary behavior. Numerical studies of the characteristic coherence time for the nonequal-time two-point function were found to be compatible with thermal behavior [17]. It would be interesting to revisit this analysis in light of our current findings.

Quenches originating or ending in the disordered phase.—Here the behavior of correlation functions of σ^x is more involved [17]. A complete summary of our results on the dynamics in this case is beyond the scope of this Letter and will be reported elsewhere [19]. On the other hand, the correlation length characterizing the stationary behavior of $\rho^{xx}(\ell, t=\infty) \sim \exp(-\ell/\xi)$ for an arbitrary quench can be cast in the simple form

$$\xi^{-1} = \theta(h-1)\theta(h_0-1)\ln(\min[h_0,h_1]) - \ln[x_+ + x_- + \theta[(h-1)(h_0-1)]\sqrt{4x_+x_-}], \tag{10}$$

where $x_\pm = \frac{1}{4}[\min(h,h^{-1})\pm 1][\min(h_0,h_0^{-1})\pm 1]$ and $h_1 = \frac{1+hh_0+\sqrt{(h^2-1)(h_0^2-1)}}{h+h_0}$. This agrees with (4) and the known results for $h_0 = 0$ and $h_0 = \infty$ [15]. Crucially, it can be proved by a direct calculation in the framework of the Toeplitz determinant approach summarized below that the result (10) agrees with the predictions of a GGE. We believe this calculation generalizes straightforwardly to nonlocal correlators in other free-fermionic theories, which suggests that the GGE correctly predicts infinite time behavior for both local and nonlocal observables in such theories.

Method I: Determinant approach.—We focus on the two-point function $\rho^{xx}(\ell,t)$, which can be written as the determinant of a $2\ell\times 2\ell$ block Toeplitz matrix T [13,17,19]. The matrix elements of T depend explicitly on the time t. In the stationary state the t dependence disappears [15] and the large-ℓ behavior can be obtained by application of the generalized Szego lemma [19], resulting in (10). The dynamics in the limit $t,\ell\to\infty$ at fixed ratio t/ℓ is much more difficult to determine, as the elements of T then depend on the matrix dimension itself and Szego's lemma does not apply. To deal with this situation we employ a method similar to [16]. In order to calculate $\ln\det|T| = \mathrm{Tr}\ln|T|$, we consider the moments of T, i.e., $\mathrm{Tr}T^{2n}$ (we find that odd moments are subleading). Calculating these moments gives

$$\mathrm{Tr}T^{2n} = \ell\int_{-\pi}^{\pi}\frac{dk}{2\pi}(\cos\Delta_k)^{2n} + \int_{-\pi}^{\pi}\frac{dk}{2\pi}\varepsilon(\ell-2|\epsilon_h'(k)|t)[1-(\cos\Delta_k)^{2n}], \tag{11}$$

where $\varepsilon(x) = x\theta(x)$. The trace of any analytic function f of T can be formally expanded in the moments. In our case we are interested in $f(x) = \ln|x|$, which is analytic in the principal strip for any $x \neq 0$. As the symbol of the block Toeplitz matrix has winding number zero about the origin and $\cos(\Delta_k)$ is always nonzero we can resum the expansion of the logarithm to obtain (4). We have generalized these results to the case of the XY spin chain in a field [19].

Method II: Form-factor approach.—This approach applies more generally to quenches in integrable (interacting) quantum field theories. We focus on the one-point function in the ordered phase. The ground state for $|h|<1$ spontaneously breaks the $\mathbb{Z}_2$ symmetry of the TFIC, resulting in an initial (ground) state of the form

$$|\Omega\rangle = \frac{1}{\sqrt{2}}[|B\rangle_R + |B\rangle_{\mathrm{NS}}], \tag{12}$$

where R and NS refer to the periodic or antiperiodic sectors of the free-fermion theory, respectively, and e.g., $|B\rangle_R = \exp(i\sum_{0<p\in R}K(p)b_p^\dagger b_{-p}^\dagger)|0\rangle_R$, where $K(p) = \tan[\Delta_p/2]$ and $b_p^\dagger$ is a fermion creation operator with momentum p. The one-point function is

$$\frac{\langle\Omega|\sigma_m^x(t)|\Omega\rangle}{\langle\Omega|\Omega\rangle} = 2\frac{{}_{\mathrm{NS}}\langle B|\sigma_m^x(t)|B\rangle_R}{{}_{\mathrm{NS}}\langle B|B\rangle_{\mathrm{NS}} + {}_R\langle B|B\rangle_R}. \tag{13}$$

Expanding the "boundary states" $|B\rangle_{R,\mathrm{NS}}$ results in a Lehmann representation for (13). Crucially, the matrix elements (form factors) of $\sigma_m^x(t)$ between multifermion Hamiltonian eigenstates are known exactly for the TFIC [22]. The main idea for evaluating the Lehmann representation is similar to the finite temperature case [23,24]. For a small quench the (total) density n_0 of fermion excitations in the initial state constitutes a small parameter. In this case one can use the $K(k)$ matrix as an expansion parameter. One then observes that the form factors appearing in the Lehmann representation are singular when momenta in the in and out states coincide. The leading (in the density n_0) contribution to the one-point function is obtained by summing all terms with the strongest singularities at a given order in the expansion in powers of K. This amounts to the exponentiation of infrared singularities. We note that just as in the finite temperature case [24] infinite volume divergences encountered in evaluating the numerator of (13) cancel against analogous divergencies in the denominator. The result of these calculations is

$$\frac{\langle\Omega|\sigma_m^x(t)|\Omega\rangle}{\langle\Omega|\Omega\rangle} \propto \exp\left[-t\int_0^\pi \frac{dk}{\pi} K^2(k) 2\epsilon'(k)\right]. \tag{14}$$

The decay rate agrees with the leading term in the expansion of (3) in powers of $K^2(k)$. The correction to the $K^2(k)$ factor in (14) are found to be $\mathcal{O}(K^6)$, again in agreement with (3). We note that (14) provides an excellent approximation to (3) as long as h, h_0 are not too close to the critical point. The two-point function can be analyzed in an analogous manner [19] and the results again agree with the appropriate expansion of (4).

Summary and discussion.—We have obtained exact analytic results for the long distance and time asymptotic behavior of one- and two-point functions of the order parameter σ^x in the TFIC after a quantum quench. We have shown that the stationary expectation value of the two-point function is not thermal, but can be described by a GGE. We have further shown that the approach to the stationary state for quenches within the ordered phase can be understood in terms of a GGE as well. Our work further demonstrates the importance of having analytic results at one's disposal when trying to draw conclusions regarding the statistical description of stationary state properties. Finally, we comment that our newly developed method based on form factors generalizes directly to integrable quenches in integrable quantum field theories. These are characterized by the requirement that the initial state is compatible with factorizable scattering. We conjecture that the stationary behavior of both local and nonlocal observables for integrable quenches in theories with purely diagonal scattering, e.g., the sine-Gordon model at a reflectionless point, can be described by an appropriate GGE.

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