Synchronized Cluster Formation in Coupled Laser Networks

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We experimentally investigate the phase dynamics of laser networks with homogenous time-delayed mutual coupling and establish the fundamental rules that govern their state of synchronization. We identified a specific substructure that imposes its synchronization state on the entire network and show that for any coupling configuration the network forms at most two synchronized clusters. Our results indicate that the synchronization state of the network is a nonlocal phenomenon and cannot be deduced by decomposing the network into smaller substructures, each with its individual synchronization state.

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It is well known that lasers can synchronize to lock their optical phases when the coupling delay time between them is relatively short (compared to the coherence time) [1–4]. Moreover, optical phase synchronization between two lasers can also occur even when the coupling delay time greatly exceeds the coherence time of the lasers themselves [5–7].

Here we experimentally present for the first time the results on the phase dynamics and synchronization states of laser networks with homogenous long time-delayed mutual coupling. We first analyze the synchronization state of simple network substructures (motifs) such as open chains and closed rings and then move on to analyze how these motifs interact with each other to form larger networks.

Experimental observations of intensity synchronization with three time-delayed coupled lasers revealed that they can be synchronized via nonsynchronized mediators [8,9]. With no time-delayed coupling, synchronization was demonstrated to extend throughout an entire network [10,11]. The relation between the symmetry of small simple coupled oscillator network motifs with short time-delayed coupling and their synchronization states was demonstrated both theoretically and experimentally [7,12]. However, the underlying interactions and synchronization within and between the basic motifs that form such networks are still an enigma, especially with long time-delayed coupling.

Figure 1 schematically shows the experimental configuration that we used for investigating phase synchronization in laser networks with time-delayed coupling and representative results. The experimental configuration includes a degenerate laser cavity [13], an arrangement for coupling different lasers, and an arrangement for detecting phase synchronization between different lasers. The degenerate cavity was composed of a Nd-Yag crystal gain medium that can support several independent laser channels, a front mirror of 10% reflectivity, a rear mirror of 40% reflectivity, a mask of an array of N apertures arranged in a desired two-dimensional lattice placed adjacent to the rear mirror, and two lenses in a $4f$ arrangement whereby any transverse electric field distribution at the mask plane was imaged onto the front mirror plane. Thus, the laser beams do not cross paths in the gain and remain uncoupled.

The coupling between the laser channels was achieved by means of mutual injection of the delayed light signal from each laser to its partner. This was accomplished by means of a focusing lens placed at a distance $f = 0.2$ m behind the rear mirror of the laser, a 50/50 beam splitter placed behind the lens, and four coupling mirrors all placed within the Rayleigh focal range of the focusing lens [4]. The reflectivities of R_1 and R_3 were 100% and those of R_2 and R_4 40%, so that each coupling mirror reflected almost equal light back at the lasers which characterizes the coupling strength and hence generates nearly uniform coupling strength (10% from R_2 and R_4 and 9% from R_1 and R_3). Each coupling mirror reflects the transverse field $E(x, y)$ from the mask plane back to obtain a reflected

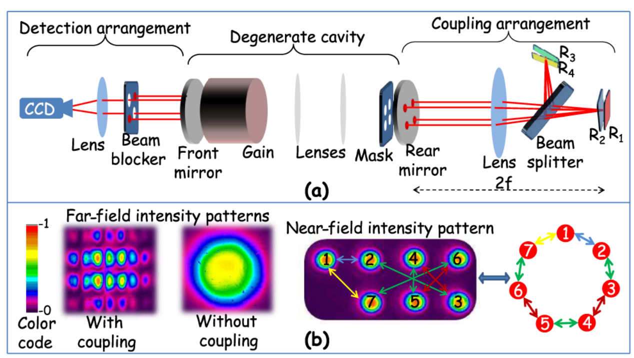

FIG. 1 (color online). Experimental configuration for coupling several independent lasers and representative results. (a) Includes a degenerate cavity, a coupling arrangement, and a detection arrangement used to detect phase synchronization between any specified lasers. (b) An example of the near-field intensity distribution of seven lasers with added arrows that denote the coupling connectivity which is rearranged for simplicity at the right. Far-field intensity distributions of all seven lasers, with and without coupling at the left.

image of $E(-(x - x_0), -(y - y_0))$, where (x_0, y_0) denote the self-reflecting point of that mirror determined by its tilting angle. By independently controlling the angular orientations of all four coupling mirrors, we realized a variety of connectivities between the lasers, whereby each mirror connects pairs of lasers that are symmetric around its self-reflecting point. Representative nearest neighbor coupling between the seven lasers is illustrated at the bottom of Fig. 1. Arrows in four different colors on the near-field intensity pattern indicate which lasers are coupled by which mirrors. The connectivity is more clearly illustrated with the sketch of the rearranged laser locations at the right.

The time delay for all 4 coupling mirrors was $\tau = 2.66$ ns, differing by less than 3 ps from one mirror to another. This delay was much longer than the coherence time of 10 ps for the individual lasers, so the coupling signal arrives long after the phase memory is lost [6]. Figure 1 also includes representative examples for the far-field intensity distributions of the seven lasers with and without the coupling. As is evident, without coupling the lasers are indeed independent and not synchronized, whereby the far-field intensity is simply a sum of the intensities from all individual lasers. This indicates that we are indeed dealing with uncorrelated independent lasers, as one would expect from our degenerate cavity. With the coupling mirrors in place, the far-field intensity pattern is characterized by a high contrast fringe pattern indicating that a high level of optical phase synchronization occurs. The phase synchronization levels can be quantified by measuring the second order correlation of the phase, namely, the fringe visibility ν [14].

We started our experiments by examining the phase dynamics of simple motifs of chains and rings of lasers with nearest neighbor time-delayed coupling. The results are presented in Fig. 2. The experimental results for phase synchronization of a chain of seven lasers are presented in Fig. 2(a). Figure 2 a_1 exemplifies the high fringe visibility ν of above 0.8 in the far-field interference pattern for any pair of odd numbered lasers in the chain (shown here with lasers 1 and 7 with $\nu = 0.91$) indicating that their phases are highly synchronized. High visibility fringes where also observed for any pair of even numbered lasers in the chain. However, the fringe visibility for nearest neighbor lasers, e.g., 4 and 5, was below 0.1, as shown in Fig. 2 a_2. These results indicate sublattice synchronization (SLS), where the chain splits into two phase synchronized clusters of odd and even lasers that are not synchronized with each other.

It is interesting to note that lasers that do not directly interact, e.g., 1 and 7, are still highly synchronized. This surprising result can be intuitively explained by noting that lasers can synchronize to the common signal they receive. Specifically, lasers 3 and 5 are synchronized because they are both coupled to the same delayed signal of laser 4, similarly to synchronization of three laser chains [8,9]. Lasers 2 and 6 are synchronized because they are both coupled to the same delayed signal of lasers 3 and 5. Finally, lasers 1 and 7 are synchronized because they are both coupled to the same delayed signal of lasers 2 and 6, and SLS emerges. Note that if the arrival times of the different coupling signals differ by more than the coherence time, SLS cannot occur.

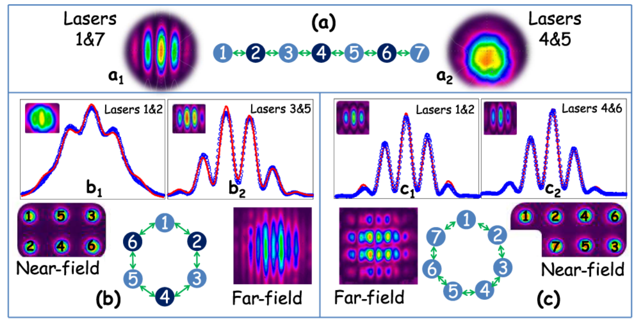

FIG. 2 (color online). Experimental phase synchronization results for chain and ring motifs. a_1 and a_2 are the far-field interference pattern of lasers 1 and 7 and lasers 4 and 5, all of which belong to a chain of seven lasers as shown in the sketch. b_1 and b_2 are the cross sections of far-field interference pattern, shown in the insets, of lasers 1 and 2 and lasers 3 and 5, all belonging to a ring of six lasers. c_1 and c_2 are the cross sections of the far-field interference pattern of lasers 1 and 2 and lasers 4 and 6, all belonging to a ring of seven lasers. Blue circles denote experimental results and the solid red curve is the fitting curve. Also shown are the near-field and far-field intensity distributions of all the lasers in each motif. Lasers denoted in the sketch with the same shade of blue belong to the same cluster.

Next we investigated the phase dynamics of a ring of lasers with nearest neighbor time-delayed coupling. The experimental results are presented in Figs. 2(b) and 2(c). Figure 2 b_1 and b_2 show the cross sections of the far-field interference patterns with a fringe visibility of $\nu = 0.1$ for a pair of one even and one odd numbered lasers, e.g., 1 and 2, and $\nu = 0.83$ for a pair of two odd numbered lasers, e.g., 3 and 5, all belonging to a ring of six lasers. Figure 2 c_1 and c_2 show the far-field interference patterns with $\nu = 0.87$ for an even and an odd numbered laser, e.g., 1 and 2, and $\nu = 0.85$ for a pair of two even numbered lasers, e.g., 4 and 6, all belonging to a ring of seven lasers. As is evident, the ring of six lasers exhibits a synchronization pattern similar to the SLS observed for open chains with $\nu > 0.8$ for any pair of two odd or two even numbered lasers, and $\nu \approx 0.1$ for any pair of odd and even numbered lasers. The ring of seven coupled lasers exhibits zero-lag synchronization (ZLS), namely, a high degree of phase synchronization with $\nu > 0.8$ between all pairs of lasers.

For a ring with an even number of six lasers, SLS is very clearly demonstrated by the far-field interference pattern shown in the lower part of Fig. 2(b). Here, high contrast interference fringes are visible only in the horizontal direction, indicating a high degree of phase synchronization only between lasers in the same row (that belong to the same sublattice). For a ring with an odd number of seven lasers, ZLS is very clearly demonstrated by the far-field interference pattern shown in the lower part of Fig. 2(c). Here, high contrast interference fringes are visible along both the horizontal and the vertical directions, indicating a high degree of phase synchronization between all lasers. Similar results of SLS were obtained for a ring of four lasers, and ZLS for rings of three and five lasers.

We turn now to explore networks constructed from several interconnected motifs with known synchronization states. Representative experimental results for networks of seven lasers with different topologies are presented in Fig. 3. Each includes a far-field interference pattern of all seven lasers that are coupled in accordance with the adjacent schematic sketch. The individual lasers were positioned in space in accordance with the near-field image at the bottom of Fig. 1. Figure 3(a) exemplifies the case of two connected previously examined motifs, i.e., one of rings of three lasers and one of six lasers, that are connected with one common edge. Here there are high contrast interference fringes along both the horizontal and the vertical directions, indicating ZLS. Figure 3(b) depicts two connected rings consisting of four and six lasers that are connected with two common edges. Here there are high contrast interference fringes only along the horizontal direction indicating SLS. These results indicate that networks maintain the SLS state only when they are composed of motifs which all share the same SLS state.

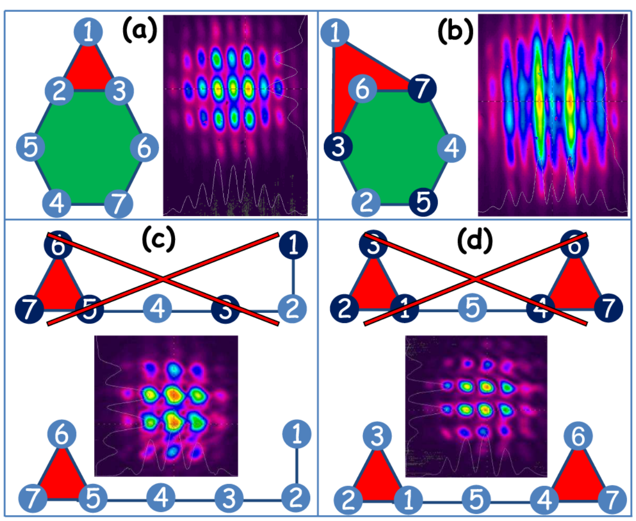

FIG. 3 (color online). Experimental phase synchronization results for four different networks composed from interconnected motifs. These show the combined far-field intensity interference pattern of all seven lasers coupled in accordance with the adjacent schematic sketch. The nodes in each sketch represent the individual lasers whose actual spatial location is in accordance with the near-field image appearing in the bottom part of Fig. 1, and the edges in each sketch represent the coupling channels. In (c) and (d) the sketches that are not consistent with the experimental data are crossed out.

Next we consider interconnected motifs that share only a single laser. Figure 3(c) shows a triangle of lasers connected to a chain of five lasers. Here, one might erroneously predict that SLS occurs with lasers 2 and 4 in one sublattice and the remaining lasers in the other sublattice, as depicted in the upper sketch. Similarly, Fig. 3(d) shows two triangles connected by a three laser chain. Again one might erroneously predict that SLS occurs with laser 5 in one sublattice and the remaining lasers in the other sublattice, as depicted in the upper sketch. But the actual experimental far-field intensity interference patterns in Figs. 3(c) and 3(d) very clearly indicate that both networks exhibit ZLS. Indeed, SLS is not a consistent solution, because it requires that laser 5 in Fig. 3(c) and lasers 1 and 4 in Fig. 3(d) must simultaneously synchronize to two different nonsynchronized signals.

The results of Fig. 3 all point to a general rule: even a single odd numbered loop breaks the consistency of SLS thereby extending its frustration throughout the entire network and forcing ZLS. Other examples of such nonlocal behavior that confirm this rule are presented in [15].

The interplay between ZLS and SLS and the emergence of at most two distinct synchronized clusters can be explained by resorting to an information mixing argument based on the adjacency matrix [15]. Based on the theory of stochastic matrices which play a central role in Markov chain processes, one can show that the mixing of information leads to a number of clusters governed by the greatest common divisor (GCD) of the loops composing the network [16,17]. For a network with mutual couplings, a loop of size two always exists. Hence, for networks with even numbered loops only the $\mathrm{GCD} = 2$ yields SLS, and for networks with at least one odd numbered loop the $\mathrm{GCD} = 1$ yields ZLS. Networks with three or more clusters may therefore only occur when coupling is unidirectional. Figure 4 shows representative examples on the temporal evolution of information mixing that lead to

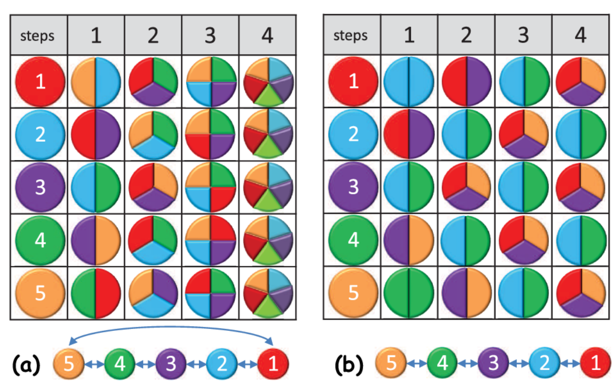

FIG. 4 (color online). Information mixing argument that leads to ZLS and SLS states. (a) is for an odd loop of five nodes; (b) for an open chain of five nodes.

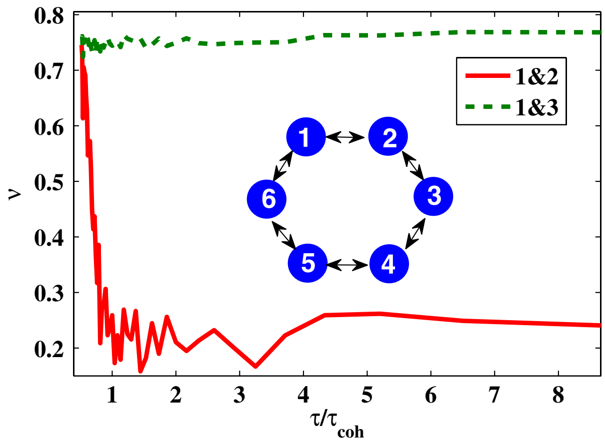

FIG. 5 (color online). Numerical integration of the Kuramoto model that describes the phase dynamics of six coupled oscillators. Calculated second order correlation function of the phase difference between two oscillators ν versus the time delay τ normalized by the coherence time τ_{coh}. The results are for different pairs of oscillators that belong to a loop of six time-delayed coupled oscillators. The dashed green curve is for oscillators 1 and 3; the solid red curve is for oscillators 1 and 2. The added sketch schematically illustrates the coupling configuration. For additional information see [15].

ZLS and SLS states [16]. Specifically, the initial condition for each node is denoted by a distinct color, the steps are rescaled with time τ, and at each time step the colors of a node denote the union of colors from its driven nodes. The colors of a specific node at any time step t indicate the set of nodes or colors at $t = 0$ which have been mixed. Figure 4(a) shows how the mixing of information results in identical nodes after 4 time steps for an odd loop of five nodes, indicating a ZLS state. Figure 4(b) shows that two distinct clusters are formed for a chain of five nodes.

To further elucidate our experimental results, we solved the Kuramoto model that describes the phase dynamics of time-delayed coupled oscillators [18]. Figure 5 shows representative results obtained from the numerical solution of the Kuramoto equations for six time-delayed coupled oscillators. As is evident, with a time-delayed coupling τ shorter than the coherence time of the oscillators τ_{coh}, nearest neighbor oscillators are synchronized. However, with $\tau > \tau_{\mathrm{coh}}$ they are not, similar to the experimental results for a ring of six lasers. We also numerically solved the Kuramoto equations for all the network configurations in our experiments, and obtained similar corresponding results [15]. In addition, we performed a numerical simulation of the full laser rate equations for coupled laser systems that includes also the full amplitude, phase, and gain dynamics of each laser [2,6]. These more accurate models yielded results essentially identical to those of the far simpler Kuramoto equations.

In addition, we numerically solved the Kuramoto equations for 18 time-delayed coupled oscillators which again confirms that only networks containing odd numbered loops exhibit ZLS (see details in [15]). The Mermin-Wagner theorem predicts that for low dimensional networks such as ours synchronization must degrade as the network size increases [10,19]. We did not observe such size related effects experimentally for networks up to seven lasers nor numerically for networks up to 18 oscillators. However, synchronization states for much larger networks would require further investigation to fully ascertain size related effects. We expect that in high dimensional networks, where the Hamilton path between all nodes is always finite, the synchronization state of the network would be governed by our rules.

To conclude, we showed that synchronized laser networks with homogeneous time-delayed mutual coupling exhibit at most two synchronized states for any coupling configuration. One is a state of zero-lag synchronization where all lasers synchronize to lock their optical phases together, occurring when the network contains a ring of odd number of lasers. The other is a state of sublattice synchronization where two synchronized clusters are formed. We believe that, although we used networks of coupled lasers, our results may be applicable for modeling other types of networks that are described by coupled oscillators [20–23].

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