# Detecting Quantum States with a Positive Wigner Function beyond Mixtures of Gaussian States

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We propose a criterion giving a sufficient condition for quantum states of a harmonic oscillator not to be expressible as a convex mixture of Gaussian states. This nontrivial property is inherent to, e.g., a single-photon state and the criterion thus allows one to reveal a signature of the state even in quantum states with a positive Wigner function. The criterion relies on directly measurable photon number probabilities and enables detection of this manifestation of a single-photon state in quantum states produced by solid-state single-photon sources in a weak coupling regime.

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Nonclassical states of a quantum harmonic oscillator [1] are an indispensable resource for quantum technology. The basic nonclassical states, squeezed states [2], are Gaussian; i.e., their Wigner function [3] is Gaussian, and they can be prepared by a unitary operation generated by a quadratic bosonic Hamiltonian. Squeezed states are a resource for the preparation of any Gaussian state and implementation of any Gaussian operation. However, a more primitive resource exists in the form of a single-photon state. If it is ideal it allows for the construction of any quantum state and more diverse quantum operations. However, a single-photon state has a non-Gaussian Wigner function, reaching negative values as a consequence of higher-order quantum nonlinearity in the state preparation. Compared to the squeezed state, the single-photon state exhibits strong anticorrelation, directly revealing its particle features. Recently, solid-state sources became very promising potential sources of single photons on demand owing to their scalability and integrability [4].

It is still a challenge for the solid-state sources to verify that the produced quantum states possess at least some of the nontrivial properties of the desired single-photon state that go beyond the framework of experimentally well managed mixtures of Gaussian states. According to definition, an arbitrarily small value of the second-order correlation function [5] can also be achieved by Gaussian states. The anticorrelation parameter $\alpha$ [6], describing the particle properties of single-photon states irrespective of losses, proves for $\alpha < 1$ only that the state is incompatible with any mixture of coherent states. The only approach used to date to verify the nontrivial non-Gaussian character of quantum states going beyond mixtures of Gaussian states utilizes the negativity of their Wigner function. The solid-state sources, however, emit light into a free space which causes a large attenuation of the produced single-photon state before it reaches the verifying detector (weak coupling regime). This strong loss between the emitter and the detector then causes the Wigner function of the state to become positive and the criterion based on its negativity cannot be used.

It is therefore imperative to develop another criterion that would also detect states with a positive Wigner function that still cannot be prepared solely using Gaussian states and operations. For this purpose it is natural to first characterize the set of all states that can be prepared in this way and that are therefore not interesting for the further development of single-photon sources. Obviously, these are the states that can be expressed as a convex mixture of Gaussian states

$$\rho_c = \int \mathcal{P}(\lambda)|\lambda\rangle\langle\lambda| d\lambda, \tag{1}$$

where $\mathcal{P}(\lambda)$ is an arbitrary normalized probability density of the parameters $\lambda$ labelling pure Gaussian states $|\lambda\rangle = S(r,\psi)D(\beta)|0\rangle$ [7], where $|0\rangle$ is the vacuum state, $S(r,\psi)$ is the squeezing operator ($r$ is the squeezing parameter and $\psi$ is the phase of the squeezing) and $D(\beta)$ is the displacement operator ($\beta$ is the complex amplitude of the displacement). Note, that any state of the form (1) can be created using only quadratic Hamiltonians, Gaussian states, and Gaussian measurements. Specifically, using Gaussian unitary operations, discarding the subsystems and homodyne detection, one can implement any operation preserving Gaussian states, the so called trace-decreasing Gaussian completely positive map [8]. If, in addition, we allow classical mixing of operations, then we can get from Gaussian input states at most a state of the form (1). The impossibility to express a state as a mixture (1) therefore reveals that its creation was assisted by tools beyond this framework. A broad variety of distributions $\mathcal{P}(\lambda)$ makes the border of the set of states (1) complex. Therefore, we focus on finding a criterion designed for the basic resource Fock states leaving general properties of the border for further research.

In this Letter, we propose a sufficient condition for a quantum state $\rho$ not to be expressible in the form (1). The criterion uses directly measurable probabilities $p_n(\rho)$ of finding the $n$th Fock state in the analyzed state. First, we answer a basic question as to how high the probability for a given Fock state can be from mixtures of Gaussian states.

 

Next, focusing on single-photon emitters we design a stronger criterion proving that a realistic single-photon source in a weak coupling regime generates states beyond the form (1).

The Fock states are a general example of highly nonclassical non-Gaussian quantum states. The $n$th Fock state $|n\rangle$ is an eigenstate of the particle number operator $N$ corresponding to the eigenvalue $n$. It was already shown that enough squeezed and displaced Gaussian state can reach arbitrarily small variance $\langle \Delta^2 N\rangle = \langle N^2\rangle - \langle N\rangle^2$, if the mean $\langle N\rangle$ is not restricted [5]. This implies that it is impossible to find the desired sufficient condition based on this variance. On the other hand, a single particle never splits at a beam splitter as proof of the fundamental integrity of the particle. In other words, using a single-particle resolving detector, the probability of detecting a single particle is unity. Instead of the variance $\langle \Delta^2 N\rangle$, a simple criterion testing incompatibility of a state $\rho$ with the decomposition (1) could be based on the probability $p_n(\rho) = \langle n|\rho|n\rangle$. Clearly, the probability satisfies $p_n(\int \mathcal{P}(\lambda)|\lambda\rangle\langle\lambda|d\lambda) = \int \mathcal{P}(\lambda)p_n(|\lambda\rangle\langle\lambda|)d\lambda$. If there is an upper bound $p_n^{\max} = p_n(|\lambda^{\max}\rangle\langle\lambda^{\max}|)$ for all $p_n(|\lambda\rangle\langle\lambda|)$, where $|\lambda^{\max}\rangle$ maximizes probability $|\langle n|\lambda\rangle|^2$, then we know that $p_n(\rho_c) \leq p_n^{\max}$ for all $\rho_c$. In other words, it is enough to find $p_n^{\max}$ by maximizing $p_n$ over pure Gaussian states. The found maximum is then a lower bound for the states which are not just a mixture of Gaussian states. Now, the main question is the following: Are the novel bounds able to detect states with positive Wigner functions incompatible with decomposition (1)?

For pure Gaussian states the probability $p_n$ reads [7]

$$p_n(|\lambda\rangle\langle\lambda|) = \left|\frac{1}{\sqrt{n!\mu}}\left(\frac{\nu}{2\mu}\right)^{n/2} H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right) \times \exp\left(-\frac{|\beta|^2}{2} + \frac{\beta^2\nu^*}{2\mu}\right)\right|^2, \quad (2)$$

where $\beta = |\beta|\exp(i\phi)$, $\mu = \cosh r$, $\nu = \sinh r \exp(i\psi)$, and $H_n$ denotes Hermite polynomials. The bound $p_n^{\max}$ can be found by optimizing (2) over $|\beta|$, $r$ and the compound phase $\Delta = \phi - \psi/2$. In the simplest case $n = 1$ the maximal probability is $p_1^{\max} = \frac{3\sqrt{3}}{4e} \approx 0.477\,889$ and it is achieved for $\Delta = 0$, $\pi$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $r = \frac{\ln 3}{2}$, $(-\frac{\ln 3}{2})$ and $|\beta|^2 = 2$. The corresponding squeezed state has position $X$ and momentum $P$ variances $\langle\Delta^2 X\rangle = 1/12$ and $\langle\Delta^2 P\rangle = 3/4$ (vacuum noise variance is $1/16$), respectively, and it is displaced along $X$ axis to the point $\langle X\rangle = \sqrt{2/3}$. The $p_n$ distribution of this optimal state is depicted in Fig. 1. We see that the probability of a single-particle state dominates which reveals that this property itself does not suffice to ensure that a state originates from a higher-order nonlinear interaction. For $n = 2$, the maximal probability is $p_2^{\max} = \frac{4}{9}\sqrt{2}(3 + 2\sqrt{3})\exp(-\frac{3}{2} - \frac{\sqrt{3}}{2}) \approx 0.381\,319$. It appears for $\Delta = 0$, $\pi$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $r = \operatorname{arccosh}\sqrt{\frac{3}{2}}$, $(-\operatorname{arccosh}\sqrt{\frac{3}{2}})$ and $|\beta|^2 = \frac{3}{2}(2 + \sqrt{3})$. It corresponds to the squeezed state with variances $\langle\Delta^2 X\rangle = (2 - \sqrt{3})/4$ and $\langle\Delta^2 P\rangle = (2 + \sqrt{3})/4$ displaced to point $\langle X\rangle = \sqrt{3/2}$.

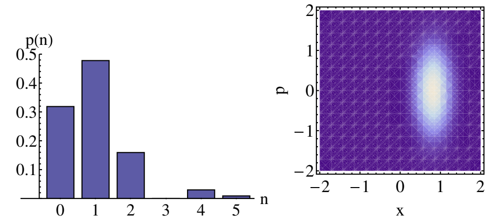


FIG. 1 (color online). Photon number distribution and Wigner function density plot of the optimal pure Gaussian state maximizing the probability of finding Fock state $|1\rangle$.

Maximization of $p_n$ for higher $n$ is a complex task that can be solved analytically only partially. Therefore, we first find $p_n^{\max}$ (up to four digits) for a few higher Fock states numerically, see Table I. For analytical optimization it is convenient to express the distribution (2) as $p_n = f g_n h_n/(2^n n!)$, where $f = \exp[-|\beta|^2(1 - \tanh r \cos\theta)]$, where $\theta = 2\Delta$, $g_n = (\tanh r)^n/\cosh r$ and $h_n = |H_n(z)|^2$, where $z = |\beta|\exp(i\theta/2)/\sqrt{\sinh(2r)}$. Maximization of $p_n$, $n \geq 1$, with respect to variables $\theta \in \langle 0, 2\pi)$, $|\beta|^2 > 0$, $r > 0$ requires us to solve the extremal equations $\partial p_n/\partial x = 0$, $x = \theta$, $|\beta|^2$, $r$ that lead to the following set of equations:

$$s_n = -|\beta|^2 \tanh r \sin\theta h_n, \quad (3)$$

$$q_n = |\beta|^2(1 - \tanh r \cos\theta)h_n, \quad (4)$$

$$q_n = \left[\frac{|\beta|^2\cos\theta}{\coth r} + n - \sinh^2(r)\right]\frac{h_n}{\cosh(2r)}, \quad (5)$$

where $s_n = -\partial h_n/\partial\theta$ and $q_n = |\beta|^2 \partial h_n/\partial|\beta|^2 = -[\tanh(2r)/2]\partial h_n/\partial r$. Combining Eqs. (4) and (5) one finds that either $h_n = 0$ which gives minimal $p_n = 0$ or

$$|\beta|^2 = \frac{n - (n+1)\tanh^2(r)}{1 + \tanh^2(r) - 2\tanh r \cos\theta}. \quad (6)$$

Note that $|\beta|^2 \geq 0$ only if $\sqrt{n/(n+1)} \geq \tanh r$. Since

$$h_n = 2\sum_{N=0}^{[n/2]}(-1)^N C^{(n,N)} T_N(\cos\theta) - C^{(n,0)}, \quad (7)$$

TABLE I. Maximal probabilities $p_n^{\max}$.

| $n$ | $p_n^{\max}$ | $\lvert\beta_{\max}\rvert^2$ | $r_{\max}$ | $n$ | $p_n^{\max}$ | $\lvert\beta_{\max}\rvert^2$ | $r_{\max}$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.4779 | 2 | 0.5493 | 5 | 0.2792 | 21.1232 | 0.8088 |
| 2 | 0.3813 | 5.5981 | 0.6584 | 6 | 0.2623 | 27.3948 | 0.8391 |
| 3 | 0.3326 | 10.1188 | 0.7243 | 7 | 0.2488 | 34.0822 | 0.8647 |
| 4 | 0.3014 | 15.3351 | 0.7717 | 8 | 0.2376 | 41.1522 | 0.8868 |

where $T_N(\cos\theta) = \cos(N\theta)$ is the Chebyshev polynomial of the first kind and

$$C^{(n,N)} = \sum_{j=0}^{[n/2]-N} \frac{(n!)^2(2|z|)^{2(n-N-2j)}}{j!(N+j)!(n-2j)!(n-2N-2j)!}, \quad (8)$$

one gets $s_n = -\partial h_n/\partial\theta = t_n \sin\theta$, where

$$t_n = 2\sum_{N=0}^{[n/2]}(-1)^N C^{(n,N)} N U_{N-1}(\cos\theta), \quad (9)$$

where $U_{N-1}(y) = (1/N)(dT_N(y)/dy)$ is the Chebyshev polynomial of the second kind. Substituting the obtained expression for $s_n$ into Eq. (3) we see that either $\theta = 0$ or

$$t_n + |\beta|^2 \tanh r h_n = 0. \quad (10)$$

Equations (4), (6), and (10) form the basis for finding one set of the candidates for the extremes of probability $p_n$. Eliminating $|\beta|^2$ from Eqs. (4) and (10) using relation (6) we arrive at the set of two polynomial equations in two variables $\cos\theta$ and $\tanh r$ whose real solutions satisfying, in addition, conditions $|\cos\theta| \le 1$ and $\sqrt{n/(n+1)} > \tanh r > 0$ comprise one set of the candidates. The other set of candidates satisfies $\theta = 0$ which simplifies the relation (6) to

$$|\beta|^2 = e^{2r}[n - \sinh^2(r)]. \quad (11)$$

The remaining equation for $\tanh r$ can be found as follows. Summing Eq. (4) with Eq. (3) multiplied by imaginary unit $i$, taking into account the relation $q_n + is_n = z(dH_n(z)/dz)H_n^*(z)$, dividing the obtained equation by $H_n^*(z) \neq 0$ and using the relation $dH_n(z)/dz = 2nH_{n-1}(z)$ we get

$$2nzH_{n-1}(z) = |\beta|^2(1 - \tanh r e^{i\theta})H_n(z). \quad (12)$$

Setting here $\theta = 0$ and substituting for $|\beta|^2$ from Eq. (11) we arrive at the remaining polynomial equation in $\tanh r$ the real roots of which fulfilling, in addition, condition $\sqrt{n/(n+1)} > \tanh r > 0$ determine the other set of candidates for extremes. Finding the roots (generally numerically), calculating $p_n$ for them with the help of Eq. (11) and picking up $r$ and $|\beta|^2$ corresponding to the largest $p_n$ we perfectly reproduce the values of $p_n^{\max}$, $|\beta_{\max}|^2$ and $r_{\max}$ from Table I. The agreement of the partially analytical results with numerical results brings us to the conjecture that for a given $n$ a global maximum of $p_n$ is achieved by a pure Gaussian state with phase $\theta = 0$, the absolute value of the displacement $\beta$ given by Eq. (11) and squeezing $r$ such that $\tanh r$ is a proper root of Eq. (12).

Previous results reveal that the global maximum of $p_n$ over all pure Gaussian states can be found analytically for $n = 0, 1, 2$. For $n \ge 3$ we have to solve the relevant equations numerically. Alternatively, we can eliminate $|\beta|$ from $p_n$ using Eq. (6), plot the obtained $p_n$ as a function of $\theta$ and $\tanh r$, and identify the maximum visually. Using the graphical method we can once again independently verify our conjecture.

As a testing physical example we can discuss a damping channel with the transmissivity $\eta$ applied to the Fock state $|n\rangle$. For $n = 1$, the obtained state $\rho_1^{(\eta)} = \eta|1\rangle\langle 1| + (1-\eta)|0\rangle\langle 0|$ has for $p_1^{\max} = 0.477\,889 < \eta < 0.5$ a positive Wigner function which is not just a mixture of Gaussian states. Thus we arrived at a very intuitive and experimentally feasible criterion detecting a relatively narrow region of states that, however, enlarges with increasing $n$. For the Fock state $|2\rangle$ propagating through the same channel the resulting state $\eta^2|2\rangle\langle 2| + (1-\eta)^2|0\rangle \times \langle 0| + 2\eta(1-\eta)|1\rangle\langle 1|$ has the same lower bound $\eta = 0.5$ for the negativity of the Wigner function. Our criterion proves that the state is not a mixture of Gaussian states already for $\eta > 0.394\,855 = (1 - \sqrt{1 - 2p_1^{\max}})/2$. This demonstrates that higher Fock states subject to loss are better for identification of the high-order nonlinearity underlying their creation.

Rather than checking the individual Fock states criteria, we combine them into a single criterion via a witness operator $\Pi$. The sufficient condition detecting states beyond the form (1) then corresponds to $P \equiv \mathrm{Tr}(\Pi\rho) > P^{\max}$. Our aim is to find as low as possible $P^{\max}$ to detect noisy versions of the typically experimentally prepared Fock state $|1\rangle$. We propose two basic witness operators $\Pi_1$ and $\Pi_a$ of the following forms. For $\Pi_1 = |1\rangle\langle 1| - \sum_{n=2}|n\rangle\langle n|$, the threshold is $P_1^{\max} = 0.349\,409$. Further improvement is reached for $\Pi_a = |1\rangle \times \langle 1| - a\sum_{n=2}|n\rangle\langle n|$, where the parameter $a > -1$ enables us to find a better bound. Maximization of $P_a = \mathrm{Tr}(\Pi_a\rho)$ over the pure Gaussian states yields

$$P_a^{\max} = \frac{\sqrt{3 + 4a(3+2a) + \sqrt{9+8a}}\exp\left(\frac{2}{1-\sqrt{9+8a}}\right)}{8\sqrt{2}(1+a)} \times (3 + 4a + \sqrt{9+8a}) - a, \quad (13)$$

which is a monotonically decreasing function of $a$ with the limit $\lim_{a\to\infty} P_a^{\max} = 0$. For mixture $\rho_1^{(\eta)}$ giving $P_a = \eta$ it is always possible to find $a$ small enough to prove that for any $0 < \eta < 1$ this state cannot be expressed as a mixture of Gaussian states, although for $\eta < 0.5$ its Wigner function is positive.

Practical applicability of the bound (13) can be demonstrated on a realistic single-photon source [4]. In the weak coupling regime the source produces a single photon to many modes from which only a single mode is effectively selected with the overall efficiency $\eta$. Assume that all imperfections in the emission and coupling can be simply modeled by a beam splitter with the transmissivity $\eta$ injected by a thermal state with mean photon number $N_d/(1-\eta)$, where $N_d$ is the mean number of thermal photons with Bose-Einstein statistics $p_m = \frac{N_d^m}{(N_d+1)^{m+1}}$. The detector efficiency and the dark-count rate can be included

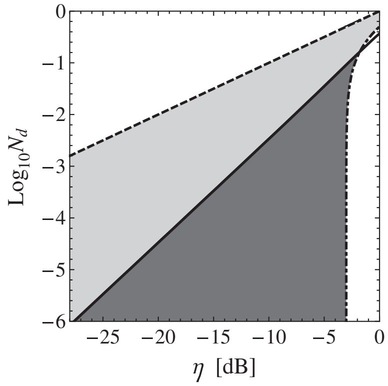


FIG. 2. Comparison between criterion based on negativity of the Wigner function (white area below dash-dotted curve) and the proposed criterion (area below solid curve) for the single photon generated by the emitter in a weak coupling regime. See text for details.

into $\eta$ and $N_d$. The Wigner function of the obtained state exhibits negativity if $N_d < \eta - 1/2$ that vanishes for $\eta < 1/2$. Consequently, for lower $\eta$ one of the significant features of the single-photon state that is the negativity of its Wigner function cannot be observed. Nevertheless, still one can test incompatibility of the state with the decomposition (1) using the proposed criterion. In the present case the probabilities of vacuum and single-photon detections read $p_0 = \frac{1-\eta+N_d}{(1+N_d)^2}$ and $p_1 = \frac{\eta+N_d(1-\eta+N_d)}{(1+N_d)^3}$. The result of our criterion after optimization of $P_a^{\max}$ is visible in Fig. 2. For states lying below the dash-dotted curve the Wigner function is negative. The dark gray area stands for the states with a positive Wigner function that are not a mixture of Gaussian states. For comparison, the threshold for the background noise degrading the states to mixtures of coherent states is $N_d < \eta$ and it is depicted by a dashed curve in Fig. 2. The states above this curve can be prepared as a mixture of coherent states. The light gray region is unknown territory where it still could be possible to develop a new stronger criterion. Figure 2 reveals a remarkable possibility to distinguish a nontrivial quantum nonlinearity in the emitter although the overall efficiency of registration is a few orders of the magnitude smaller. Like in the case of single-photon states the proposed criterion can be targeted on superposition of coherent states [9]. When prepared experimentally the states are practically very close to squeezed Fock states and it is just needed to first apply an optimal resqueezing operation [10] on the Wigner function estimated from the homodyne tomography and then to use the criterion.

In conclusion, we proposed a novel criterion detecting quantum states with a positive Wigner function that cannot be expressed as a convex mixture of Gaussian states. We have analyzed prospective applications of this criterion for two different sources of heralded single photons involving photon subtraction [9] and photon counting on a two-mode squeezed state [11]. For low squeezing, the criterion detects the states incompatible with a mixture of Gaussian states substantially better than the negativity of the Wigner function (measured by a homodyne detector), tolerating a less efficient and noisier heralding detector. It can point to a promising direction in an experiment which can enable the preparation of a single photon with a negative Wigner function after overcoming technical limitations. The question of further applications for these highly nonclassical states with a positive Wigner function remains open. Nevertheless, we have found that squeezing can be concentrated from such imperfect single photons. Similarly, for the solid-state single-photon sources the criterion can complement the test of an anticorrelation parameter utilizing the same experimental setup and proves a stronger statement, that the produced states are even incompatible with any state preparation based on a mixture of Gaussian states. If this criterion detects the property for an emitter, other techniques can be applied to further improve the emitter's performance. In order to achieve a strong coupling regime, sophisticated metallic nanowires can later be used to enhance photon harvesting into a well-localized mode [12]. Once a large enough portion of the emitted signal is captured, a negative Wigner function can be observed proving the presence of a stronger nonclassical property of the resource single-photon state in the generated state.

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