Gyrokinetic Fokker-Planck Collision Operator

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The gyrokinetic linearized exact Fokker-Planck collision operator is obtained in a form suitable for plasma gyrokinetic equations, for arbitrary mass ratio. The linearized Fokker-Planck operator includes both the test-particle and field-particle contributions, and automatically conserves particles, momentum, and energy, while ensuring non-negative entropy production. Finite gyroradius effects in both field-particle and test-particle terms are evaluated. When implemented in gyrokinetic simulations, these effects can be precomputed. The field-particle operator at each time step requires the evaluation of a single two-dimensional integral, and is not only more accurate, but appears to be less expensive to evaluate than conserving model operators.

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Collisions play an important role in plasma turbulence and can strongly influence turbulent transport in magnetic fusion experiments. Within terms of the order of the inverse Coulomb logarithm, large angle Coulomb collisions and long time correlations, which relax on the short time scale of plasma oscillations, can be neglected [1]. Considering small-angle Coulomb collisions results in the Fokker-Planck collision operator [1,2]. The linearized full collision operator includes the test-particle and field-particle operators, and satisfies Boltzmann's H theorem and the conservation of particles, momentum, and energy [3,4]. The field-particle operator, which involves the Rosenbluth potentials of the non-Maxwellian distribution, has generally been considered intractable. It is often approximated by ''conserving'' terms that restore the conservation of low-order moments to the test-particle operator.

In situations involving frequencies much less than the gyrofrequency, where the equilibrium varies weakly on the gyroradius (Larmor radius) spatial scale, the Fokker-Planck equation can be averaged over the particle gyration, reducing its dimensionality from six to five phase space dimensions. When finite gyroradius effects are accounted for, this transformation results in the gyrokinetic equation [5–7]. This equation describes low-frequency turbulence driven by gradients of the plasma density, temperature, and flows across the magnetic field, in magnetically confined laboratory plasmas. However, a gyrokinetic version of the exact Fokker-Planck collision operator for this equation has not been developed.

Significant effort has been made to construct a model gyrokinetic operator which includes the effect of finite gyroradius [5], and conserves particles, momentum, and energy. The test-particle terms of this operator, including energy scattering, were implemented in the gyrokinetic code GS2 [8], and shown to strongly damp short wavelength trapped electron modes. Most recently, new conserving terms were formulated to ensure non-negative entropy production [9,10]. Upon implementation in the GS2 code [11], this model operator yields a similar damping of short wavelength entropy modes, in general providing physical dissipation at short wavelengths. Quantitative accuracy is suggested in comparisons with analytic theory for the damping of electromagnetic waves by resistivity [11]. However, calculation of the classical collisional ion heat transport across the magnetic field, using the same model operator, reveals a $\sim$50% discrepancy [10] relative to the linearized full Fokker-Planck operator. This shows that present model operators still do not have all of the properties of the full collision operator.

In this Letter, we present the gyrokinetic linearized full Fokker-Planck collision operator for arbitrary mass ratio, including both the test-particle and field-particle contributions. The gyrokinetic operator we obtain accounts for finite gyroradius effects in both the test-particle and field-particle terms within the usual gyrokinetic ordering. The gyrophase averages resulting from the field-particle operator are independent of the guiding-center (gyroaveraged) distribution, and can therefore be precomputed. Once precomputed, these gyroaverages can be efficiently reused in successive time steps in gyrokinetic simulations. Further, the field-particle operator at each time step is reduced to the evaluation of a single two-dimensional velocity integral over the evolving guiding-center distribution. In contrast, recent model operators typically require the evaluation of several more complex integrals for the conserved moments. Thus the exact operator is not only more accurate, but much less expensive to evaluate than conserving model operators.

For completeness, the linearized full collision operator is first derived in a symmetric form, utilizing Rosenbluth potentials. This simplifies both the test-particle and field-particle terms. The Fokker-Planck collision operator gives the time rate of change in the distribution function f_a of species a due to Coulomb collisions with particles of species b [1,3], $(\partial f_a/\partial t)_c = \sum_b C(f_a, f_b)$. It is often convenient to express the operator in terms of the Rosenbluth

potentials $G(\boldsymbol{v}) = \int d^3v' f'_b u$ and $H(\boldsymbol{v}) = \int d^3v' f'_b/u$, where $f = f(\boldsymbol{v})$, $f' = f(\boldsymbol{v}')$, and $u = |\boldsymbol{v} - \boldsymbol{v}'|$ is the relative velocity [2,12]. Employing the properties of the Rosenbluth potentials $\nabla_v^2 H = -4\pi f_b$ and $\nabla_v^2 G = 2H$, the full Fokker-Planck collision operator may be written in the symmetric form [5]

$$C(f_a, f_b)/\Gamma = \frac{1}{2}\frac{\partial^2 G}{\partial v_i \partial v_j}\frac{\partial^2 f_a}{\partial v_i \partial v_j} + 4\pi\frac{m_a}{m_b} f_a f_b + \left(1 - \frac{m_a}{m_b}\right)\frac{\partial H}{\partial v_i}\frac{\partial f_a}{\partial v_i}, \quad (1)$$

where $\Gamma = 4\pi Z_a^2 Z_b^2 e^4 \ln\Lambda/m_a^2$ and the sum over repeated indices i and j is understood. The form Eq. (1) is symmetric in the sense that exchanging the distribution function and Rosenbluth potentials results in the same differential form. For small departures from thermodynamic equilibrium, the distribution functions for all species are nearly Maxwellian, $f = f_0 + f_1$, where $f_0 = (n/\pi^{3/2} v_T^3)\exp(-v^2/v_T^2)$ is Maxwellian for a species of density n, the perturbed distribution $f_1 \ll f_0$, and $v_T = \sqrt{2T/m}$ is the thermal speed. It follows from Eq. (1) that the *linearized* full Fokker-Planck collison operator consists of the sum of the test-particle operator [5]

$$C(f_{a1}, f_{b0})/\Gamma = \frac{1}{2}\frac{\partial^2 G_0}{\partial v_i \partial v_j}\frac{\partial^2 f_{a1}}{\partial v_i \partial v_j} + 4\pi\frac{m_a}{m_b} f_{b0} f_{a1} + \left(1 - \frac{m_a}{m_b}\right)\frac{\partial H_0}{\partial v_i}\frac{\partial f_{a1}}{\partial v_i}, \quad (2)$$

and the field-particle operator

$$C(f_{a0}, f_{b1})/\Gamma = \frac{1}{2}\frac{\partial^2 G_1}{\partial v_i \partial v_j}\frac{\partial^2 f_{a0}}{\partial v_i \partial v_j} + 4\pi\frac{m_a}{m_b} f_{a0} f_{b1} + \left(1 - \frac{m_a}{m_b}\right)\frac{\partial H_1}{\partial v_i}\frac{\partial f_{a0}}{\partial v_i}, \quad (3)$$

where the Rosenbluth potentials of Maxwellian distribution $G_0(v) = \int d^3v' f'_{b0} u$ and $H_0(v) = \int d^3v' f'_{b0}/u$ are isotropic in velocity space; $G_1(\boldsymbol{v}) = \int d^3v' f'_{b1} u$ and $H_1(\boldsymbol{v}) = \int d^3v' f'_{b1}/u$ evaluated using the perturbed distribution are in general anisotropic in velocity space.

It is convenient to write the collision operator in terms of spherical velocity variables $v = |\boldsymbol{v}|$, pitch angle $\xi = v_\parallel/v$, and gyrophase ϕ defined with respect to the magnetic field $\boldsymbol{B} = B\boldsymbol{b}$, where the velocity $\boldsymbol{v} = v_\perp(\hat{e}_1\cos\phi + \hat{e}_2\sin\phi) + v_\parallel \boldsymbol{b}$ and $v_\perp = v\sqrt{1-\xi^2}$. Then the test-particle operator Eq. (2) can be rewritten as the sum of the Lorentz operator (pitch-angle scattering) [5]

$$C_L(f_{a1}, f_{b0}) = \frac{\nu_D}{2}\left[\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi} + \frac{1}{1-\xi^2}\frac{\partial^2}{\partial\phi^2}\right] f_{a1} \quad (4)$$

and energy scattering

$$C_E(f_{a1}, f_{b0}) = \nu_\parallel \frac{v^2}{2}\frac{\partial^2 f_{a1}}{\partial v^2} - \nu_s\left(1 - \frac{m_a}{m_b}\right) v\frac{\partial f_{a1}}{\partial v} + \nu_D v\frac{\partial f_{a1}}{\partial v} + 4\pi\Gamma\frac{m_a}{m_b} f_{b0} f_{a1}, \quad (5)$$

where the frequencies $\nu_D = (\Gamma/v^3)dG_0/dv$, $\nu_\parallel = (\Gamma/v^2)d^2G_0/dv^2$, $\nu_s = -(\Gamma/v)dH_0/dv$. For a Maxwellian background f_{b0}, the frequencies can be expressed as [4,10] $\nu_\parallel = 2(\Gamma n_b/v^3)\Psi(x)$, $\nu_D = (\Gamma n_b/v^3)\mathrm{erf}(x) - \nu_\parallel/2$, $\nu_s = x^2\nu_\parallel$, where $\mathrm{erf}(x)$ is the error function, $\Psi(x) = [\mathrm{erf}(x) - x\mathrm{erf}'(x)]/(2x^2)$ is the Chandrasekhar function, and $x = v/v_{Tb}$.

Because of the symmetry in Eq. (1), the differential form of the field-particle operator is similar to the test-particle operator and the Lorentz operator on G_1 can be removed by employing the property $\nabla_v^2 G_1 = 2H_1$. Thus the field-particle operator Eq. (3) can be rewritten as

$$C(f_{a0}, f_{b1})/\Gamma = \frac{1}{2}\frac{d^2 f_{a0}}{dv^2}\frac{\partial^2 G_1}{\partial v^2} + \frac{1}{2v}\frac{df_{a0}}{dv}\left(2H_1 - \frac{\partial^2 G_1}{\partial v^2}\right) + \left(1 - \frac{m_a}{m_b}\right)\frac{\partial H_1}{\partial v}\frac{df_{a0}}{dv} + 4\pi\frac{m_a}{m_b} f_{a0} f_{b1}. \quad (6)$$

For the Maxwellian f_{a0} with $v_{Ta} = \sqrt{2T_a/m_a}$, the field-particle operator Eq. (6) becomes [4,13]

$$C(f_{a0}, f_{b1})/(\Gamma f_{a0}) = \frac{2v^2}{v_{Ta}^4}\frac{\partial^2 G_1}{\partial v^2} - \frac{2v}{v_{Ta}^2}\left(1 - \frac{m_a}{m_b}\right)\frac{\partial H_1}{\partial v} - \frac{2}{v_{Ta}^2} H_1 + 4\pi\frac{m_a}{m_b} f_{b1}. \quad (7)$$

The collision operator in the gyrokinetic equation is obtained via the transformation from guiding-center to particle coordinates, application of the collision operator, and transformation back to guiding-center coordinates, followed by the average over the gyroangle [5,6,9]. Thus the gyrokinetic linearized full Fokker-Planck collision operator consists of the gyrokinetic test-particle operator

$$C^{\mathrm{gk}}(h_a, f_{b0}) = \langle e^{-iL_a} C(h_a e^{iL_a}, f_{b0})\rangle, \quad (8)$$

and the gyrokinetic field-particle operator

$$C^{\mathrm{gk}}(f_{a0}, h_b) = \langle e^{-iL_a} C(f_{a0}, h_b e^{iL_b})\rangle, \quad (9)$$

with the gyrophase average $\langle\cdots\rangle = \oint d\phi/2\pi$. The perturbed guiding-center distribution h is independent of the gyroangle, $L_s = (\boldsymbol{v}\times\boldsymbol{b})\cdot\boldsymbol{k}_\perp/\Omega_s$ and $\Omega_s = Z_s eB/m_s c$ for species s. The binormal basis vector $\hat{e}_1$ may be chosen in the direction of the perpendicular wave number so that $\boldsymbol{k}_\perp = k\hat{e}_1$. Then $L_s = k\sin\phi v_\perp/\Omega_s$ and $L'_s = k\sin\phi' v'_\perp/\Omega_s$.

It is well known that the gyroaverage of the test-particle operator results in finite gyroradius terms that are proportional to k^2v^2/Ω^2 [5,9,10]. Inserting the test-particle

operator into the formula Eq. (8), the gyroaverage of the Lorentz operator Eq. (4) yields [9,10]

$$C_L^{\rm gk}(h_a, f_{b0}) = \frac{\nu_D}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}h_a - \nu_D(1+\xi^2)\frac{k^2v^2}{4\Omega_a^2}h_a, \quad (10)$$

and the gyroaverage of energy scattering Eq. (5) gives

$$C_E^{\rm gk}(h_a, f_{b0}) = \nu_\parallel\frac{v^2}{2}\frac{\partial^2 h_a}{\partial v^2} - \nu_s\left(1-\frac{m_a}{m_b}\right)v\frac{\partial h_a}{\partial v} + \nu_D v\frac{\partial h_a}{\partial v} + 4\pi\Gamma\frac{m_a}{m_b}f_{b0}h_a - \nu_\parallel(1-\xi^2)\frac{k^2v^2}{4\Omega_a^2}h_a. \quad (11)$$

For the calculations of the field-particle operator, it is convenient to normalize the velocities to the thermal speed, i.e., $v/v_{Ta} \to v$ and the wave number k to the thermal gyroradius $\rho_a = v_{Ta}/\Omega_a$, i.e., $k\rho_a \to k$, then $L_a = L_b/\alpha = kv_\perp\sin\phi$, where $\alpha = \Omega_a/\Omega_b = Z_a m_b/(m_a Z_b)$. Inserting the field-particle operator Eq. (7) into the formula Eq. (9), the gyrokinetic field-particle operator then takes the form

$$C^{\rm gk}(f_{a0}, h_b)/(\Gamma f_{a0}) = \langle e^{-iL_a}P\rangle + 4\pi\frac{m_a}{m_b}h_b\langle e^{i(L_b-L_a)}\rangle, \quad (12)$$

where the normalized quantity

$$P = 2v^2\frac{\partial^2 G_1}{\partial v^2} - 2H_1 - 2\left(1-\frac{m_a}{m_b}\right)v\frac{\partial H_1}{\partial v}, \quad (13)$$

the Rosenbluth potentials $G_1 = \int d^3v' h_b' \exp(iL_b')u$ and $H_1 = \int d^3v' h_b' \exp(iL_b')/u$. The integral representation of Bessel functions gives

$$\langle e^{i(L_b-L_a)}\rangle = J_0[kv_\perp(1-\alpha)]. \quad (14)$$

Since the guiding-center distribution h' is independent of gyroangle ϕ', we have

$$\langle e^{-iL_a}P\rangle = 2\pi\int d^2v' h_b'(v', \xi')I(v, \xi, v', \xi', k), \quad (15)$$

where the normalized gyrophase integral

$$I(v, \xi, v', \xi', k) = \oint\frac{d\phi}{2\pi}\oint\frac{d\phi'}{2\pi}\exp(iL_b' - iL_a)U, \quad (16)$$

$$U = 2v^2\frac{\partial^2 u}{\partial v^2} - \frac{2}{u} - 2\left(1-\frac{m_a}{m_b}\right)v\frac{\partial}{\partial v}\frac{1}{u}, \quad (17)$$

$$u = [v^2 + v'^2 - 2v_\perp v_\perp'\cos(\phi-\phi') - 2v_\parallel v_\parallel']^{1/2}. \quad (18)$$

The velocity derivatives in Eq. (17) can be expressed as $2v^2\partial^2 u/\partial v^2 = (v^2+v'^2)/u - u/2 - (v^2-v'^2)^2/(2u^3)$ and $2v\partial(1/u)\partial v = -1/u - (v^2-v'^2)/u^3$, so that

$$U = \left(v^2 + v'^2 - 1 - \frac{m_a}{m_b}\right)\frac{1}{u} - \frac{u}{2} + \left[\left(1-\frac{m_a}{m_b}\right)(v^2-v'^2) - \frac{(v^2-v'^2)^2}{2}\right]\frac{1}{u^3}. \quad (19)$$

The finite gyroradius effects described by I in gyrophase integral Eq. (16) can be precomputed independently of the evolving distribution function h', given the velocity grid (v, ξ, v', ξ') and wave number k. Thus the field-particle operator at each time step in gyrokinetic simulations is reduced to the evaluation of a single two-dimensional velocity integral, Eq. (15). Note that the quantity U defined in Eq. (17) depends only on the difference $\phi - \phi'$ through the relative velocity u which is periodic and even in $\phi - \phi'$. In the limit $k = 0$, the gyrophase integral Eq. (16) reduces to

$$I_0 = I(v, \xi, v', \xi', k=0) = \oint\frac{d\phi'}{2\pi}U(\phi'-\phi) \quad (20)$$

The relative velocity given by Eq. (18) can be rewritten as $u = \lambda\sqrt{1-\kappa^2\sin^2\theta}$, where $\theta = (\pi+\phi-\phi')/2$, $\lambda^2 = (v_\perp + v_\perp')^2 + (v_\parallel - v_\parallel')^2$, and $\kappa^2 = 4v_\perp v_\perp'/\lambda^2$. Thus I_0, the gyrophase integral I evaluated at wave number $k = 0$, may be expressed in terms of complete elliptic integrals K and E [1,14] using the results $\oint d\phi'/u = 4K(\kappa)/\lambda$, $\oint d\phi' u = 4\lambda E(\kappa)$, $\oint d\phi'/u^3 = 4E(\kappa)/[\lambda^3(1-\kappa^2)]$.

The periodic function U may be decomposed into a Fourier series $U(\phi-\phi') = \sum_n c_n e^{in(\phi-\phi')}$ where the coefficient c_0 is equal to I_0. Using the integral representation of Bessel functions $\oint d\phi\exp(ikv_\perp\sin\phi - in\phi) = 2\pi J_n(kv_\perp)$, the gyrophase integral Eq. (16) then becomes

$$I = \sum_n c_n J_n(kv_\perp)J_n(kv_\perp'\alpha). \quad (21)$$

Since the Fourier coefficients c_n are real for even functions, the gyrophase integral Eq. (16) is real and can be rewritten as

$$I = \oint\frac{d\phi}{2\pi}\oint\frac{d\phi'}{2\pi}\cos(kv_\perp'\alpha\sin\phi' - kv_\perp\sin\phi)U. \quad (22)$$

The gyrophase integral in the form of Eq. (22) can be accurately precomputed numerically using an adaptive multidimensional integration algorithm [15]. For large $kv_\perp$ and $kv_\perp'$, the gyrophase integral can be asymptotically approximated using the saddle point method as well. For unlike particle collisions such as electron-ion collisions, the collision operators can be simplified using a large mass ratio $m_i/m_e \gg 1$ expansion [2]. However, for like particle and ion-impurity collisions, the exact operator is necessary.

Figure 1 shows the typical wave number dependence of the gyrophase integral I/I_0 versus $k\rho$ for self-collisions. Here $\rho = v_T/\Omega$ is the thermal gyroradius. Since the integrand in the gyrophase integral becomes oscillatory for large k values, the wave number dependence is

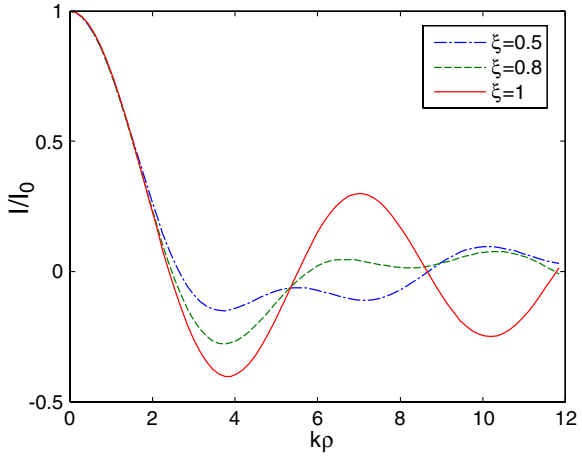

FIG. 1 (color online). Typical wave number dependence of gyrophase integral I/I_0 for self-collisions. Here $v' = 1$, $\xi' = 0$, $v = 0.5$, $\xi = 1$ (solid line), $\xi = 0.8$ (dashed line), $\xi = 0.5$ (dash-dotted line).

characterized by oscillatory decay and the largest integral value is at $k = 0$. For $v_\perp = 0$ or $v'_\perp = 0$ the gyrophase integral becomes $I/I_0 = J_0(kv'_\perp)$ or $J_0(kv_\perp)$. Thus in Fig. 1 for the case $\xi = 1$ with $v_\perp = 0$ and $v'_\perp = 1$, the zeros of gyrophase integral $I/I_0 = J_0(k)$ occur at $k \approx 2.4, 5.5, \ldots$ the roots of Bessel function of order $n = 0$, verifying the accuracy of numerical integration.

The Debye length λ_D characterizes charge shielding, and effectively limits the range of interaction between charges, limiting the maximum impact parameter for small-angle deflections in the plasma [1]. This excludes the value $u = 0$ for the case $\boldsymbol{v} = \boldsymbol{v}'$ from the calculation, since the classical distance of closest approach $e^2/(mu^2/2)$ should be smaller than the Debye length. If the function $U(\phi - \phi')$ is smooth with no jumps, as is the case when $\boldsymbol{v}$ and $\boldsymbol{v}'$ differ significantly, then the Fourier series converges quickly so that the gyrophase integral may be approximated by a small number of terms in Eq. (21). Thus in Fig. 1 for the case $\xi = 0.8$ with $v_\perp = 0.3$ and $v'_\perp = 1$, the gyrophase integral $I/I_0 \approx J_0(kv_\perp)J_0(kv'_\perp)$. In gyrokinetic simulations, the full expression in Eq. (22) can be readily precomputed. The gyrokinetic linearized Fokker-Planck operator is finally

$$\left(\frac{\partial h_a}{\partial t}\right)_c = C_L^{\mathrm{gk}}(h_a, f_{b0}) + C_E^{\mathrm{gk}}(h_a, f_{b0}) + \nu_a \frac{2}{\sqrt{\pi}} \left\{ \frac{2m_a}{m_b} h_b J_0\left(\frac{kv_\perp}{\Omega_a}\left[1 - \frac{\Omega_a}{\Omega_b}\right]\right) + \int d^2v' h'_b(v', \xi') I(v, \xi, v', \xi', k) \right\} \exp(-v^2/v_{Ta}^2), \tag{23}$$

where the frequency $\nu_a = n_a \Gamma / v_{Ta}^3$ and the sum over all species b including $b = a$ is understood.

In summary, the gyrokinetic linearized full Fokker-Planck collision operator for arbitrary mass ratio has been obtained, including both field-particle and test-particle contributions. Finite gyroradius effects are included in both field-particle and test-particle terms. These effects were obtained via the transformation from guiding-center to particle coordinates, application of the collision operator, and transformation back to guiding-center coordinates, averaging over the gyroangle. The full operator automatically conserves particles, momentum, and energy, while ensuring positive entropy production. We find that the finite gyroradius effects can be precomputed independently of the evolving distribution function, leaving a single two-dimensional velocity integral over the evolving distribution function. Accordingly, this operator is both more accurate and appears to be less computationally expensive than model operators which approximate the field-particle terms with conserving terms.

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