Taming Multiparticle Entanglement

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We present an approach to characterize genuine multiparticle entanglement by using appropriate approximations in the space of quantum states. This leads to a criterion for entanglement which can easily be calculated by using semidefinite programing and improves all existing approaches significantly. Experimentally, it can also be evaluated when only some observables are measured. Furthermore, it results in a computable entanglement monotone for genuine multiparticle entanglement. Based on this, we develop an analytical approach for the entanglement detection in cluster states, leading to an exponential improvement compared with existing schemes.

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Introduction.—The characterization of multiparticle quantum correlations is relevant for many physical systems like atoms in optical lattices, superconducting qubits, or nitrogen-vacancy centers in diamond, to name only some recent examples [1]. In the field of quantum information, multiparticle entanglement is viewed as a resource, enabling tasks like measurement-based quantum computation [2] or high-precision metrology [3]. In spite of many efforts, the characterization of these correlations turns out to be difficult. Especially genuine multipartite entanglement, which is most important from the experimental point of view, remains unruly, and only scattered results concerning its characterization are known [4–7].

In this Letter, we derive a general method to characterize genuine multiparticle entanglement using suitable relaxations. This relaxed problem turns out to be good-natured, can be tackled with different methods, and results in a criterion that can be considered as a generalization of the Peres-Horodecki criterion [8] to the multipartite case. The goal of our work is twofold. First, we present powerful criteria for genuine multiparticle entanglement, which can be efficiently evaluated by using semidefinite programing and improve existing conditions significantly. They work for multiqubit, continuous-variable, or hybrid systems and can be evaluated, even if the mean values of only a few observables are known. Furthermore, they lead to a computable entanglement monotone for genuine multiparticle entanglement.

Second, our method allows us to analytically derive entanglement conditions for the family of cluster states [9], which are important states for tasks like measurement-based quantum computation. The sensitivity of these conditions improves exponentially with the number of qubits, which is an exponential gain compared with the existing conditions. As a side product of our investigations, we will also estimate the volume of the set of genuinely multipartite entangled states and gain insight into the geometrical form of the set of biseparable states.

Situation.—We start by considering a three-particle quantum state ϱ. This state is separable with respect to some bipartition, say, $A|BC$, if it is a mixture of product states with respect to this bipartition: $\varrho = \sum_k q_k |\phi_A^k\rangle\langle\phi_A^k| \otimes |\psi_{BC}^k\rangle\langle\psi_{BC}^k|$, where the q_k form a probability distribution. We denote these states by $\varrho_{A|BC}^{\text{sep}}$. Similarly, we can define the separable states for the two other possible bipartitions $\varrho_{B|AC}^{\text{sep}}$ and $\varrho_{C|AB}^{\text{sep}}$.

Then, a state is called *biseparable* if it can be written as a mixture of states which are separable with respect to different bipartitions [4]. That is, one has

$$\varrho^{\text{bs}} = p_1 \varrho_{A|BC}^{\text{sep}} + p_2 \varrho_{B|AC}^{\text{sep}} + p_3 \varrho_{C|AB}^{\text{sep}}. \quad (1)$$

On the other hand, a state that is not biseparable is called *genuinely multipartite entangled*. Whenever we talk about multipartite entangled states in the following, we refer to genuinely multipartite entangled states.

To characterize multipartite entanglement, we apply the method illustrated by Fig. 1. Instead of states like $\varrho_{A|BC}^{\text{sep}}$ that are separable with respect to a fixed bipartition, we consider a larger set of states, which can be more easily characterized. For instance, for the bipartition $A|BC$ we may consider states which have a positive partial transpose (PPT) [10]. It is well known that separable states are also PPT [8]. We denote such states by $\varrho_{A|BC}^{\text{ppt}}$ (and analogously for the other bipartitions).

Thus, we ask whether a state can be written as

$$\varrho^{\text{pmix}} = p_1 \varrho_{A|BC}^{\text{ppt}} + p_2 \varrho_{B|AC}^{\text{ppt}} + p_3 \varrho_{C|AB}^{\text{ppt}}. \quad (2)$$

We call states of this form *PPT mixtures*. Clearly, any biseparable state is a PPT mixture, so proving that a state is no PPT mixture implies genuine multipartite entanglement. There are examples of states which are PPT with respect to any bipartition but nevertheless multipartite

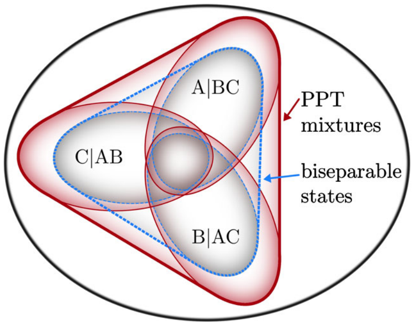

FIG. 1 (color online). For three qubits, there are three convex sets of states that are separable with respect to a fixed bipartition, namely, the bipartitions $A|BC$, $B|AC$, and $C|AB$ (blue, dashed lines). Their convex hull (thick blue, dashed line) is the set of biseparable states. Each of the three sets is contained within the set of states that are PPT with respect to the corresponding bipartition (red, solid lines). Their convex hull forms the set of PPT mixtures (thick red, solid line).

entangled [11]. Hence, not all multipartite entangled states can be detected in this way, but, as we will see, often the set of PPT mixtures is a very good approximation to the set of biseparable states. Finally, note that all definitions can be extended to N particles. Also, one may use other relaxations of bipartite separability, e.g., apply the criterion of Doherty, Parrilo, and Spedalieri [12].

The advantage of considering PPT mixtures instead of biseparable states is that the set of PPT mixtures can be fully characterized with the method of linear semidefinite programing (SDP) [13]—a standard problem of constrained convex optimization theory. Moreover, PPT mixtures can also be characterized analytically.

Characterization via entanglement witnesses.—An *entanglement witness* is an observable W that is non-negative on all biseparable states but has a negative expectation value on at least one entangled state. Let us first consider two particles A and B. Then a *decomposable witness* is a witness W that can be written as $W = P + Q^{T_A}$, where P and Q have no negative eigenvalues (they are positive semidefinite: $P, Q \geq 0$) and T_A is the partial transpose with respect to A [14].

For more than two particles, we call a witness W *fully decomposable* if, for every subset M of all systems, it is decomposable with respect to the bipartition given by M and its complement $\overline{M}$. This means there exist positive semidefinite operators P_M and Q_M such that

$$\text{for all } M\text{: } W = P_M + Q_M^{T_M}. \tag{3}$$

This observable is positive on all PPT mixtures, as it is non-negative on all states which are PPT with respect to some bipartition. But also the converse holds.

Observation.—If ϱ is not a PPT mixture, then there exists a fully decomposable witness W that detects ϱ.

Proof.—The set of PPT mixtures is convex and compact. Therefore, for any state outside of it, there is a witness that is positive on all PPT mixtures. Furthermore, positivity on all states that are PPT with respect to a fixed (but arbitrary) bipartition implies that the witness is decomposable with respect to this fixed (but arbitrary) bipartition [14]. Thus, $W = P_M + Q_M^{T_M}$ for any M. ■

Practical evaluation.—To find a fully decomposable witness for a given state, the convex optimization technique SDP becomes important, since it allows us to optimize over all fully decomposable witnesses. Given a multipartite state ϱ, the search is given by

$$\min \mathrm{Tr}(W\varrho) \tag{4}$$

such that $\mathrm{Tr}(W) = 1$ and for all M:

$$W = P_M + Q_M^{T_M}, \quad Q_M \geq 0, \quad P_M \geq 0.$$

The free parameters are given by W and the operators P_M for every subset M. If the minimum in Eq. (4) is negative, ϱ is not a PPT mixture and hence is genuinely multipartite entangled. The operator W for which the negative minimum is obtained is a fully decomposable witness. Note that, due to $X^{T_M} = (X^T)^{T_{\overline{M}}}$ and $X \geq 0 \Leftrightarrow X^T \geq 0$, a witness that is decomposable with respect to M is also decomposable with respect to $\overline{M}$. Thus, one needs to check only half of all subsets in practice.

Equation (4) has the form of a semidefinite program [13]. In contrast to usual optimization problems, global optimality of an SDP can be certified and the solution can efficiently be computed via interior-point methods. In practice, Eq. (4) can be solved with few lines of code, by using, e.g., the parser YALMIP [15] and, as solvers, SEDUMI [16] or SDPT3 [17]. Our implementation in MATLAB named PPTMIXER can be found online [18].

Let us discuss two variations of Eq. (4). First, in order to reduce the number of free parameters, one can restrict oneself to witnesses W that obey $W^{T_M} \geq 0$ for all M, i.e., $P_M = 0$ for all M. In the following, we will call these witnesses *fully PPT witnesses*. For bipartite systems, decomposable witnesses and fully PPT witnesses detect the same states. For the multipartite case, fully PPT witnesses are not as good as fully decomposable witnesses, but they are easier to characterize.

Second, this SDP can also be modified to account for the case that, instead of a full tomography, only a restricted set of observables is measured. Let $\mathcal{O} = \{O_1, \ldots, O_k\}$ be such a set of observables. Then, adding $W = \sum_{i=1}^k \lambda_i O_i$ to the constraints in Eq. (4) results in an SDP that searches for witnesses which can be evaluated by knowing these observables. Note that for this program the free parameters are given by the real numbers λ_i, and their number might be considerably smaller than in Eq. (4). If the minimum in Eq. (4) is non-negative, there exists a PPT mixture with expectation values $\langle O_i \rangle$. However, one may then add further observables to $\mathcal{O}$ and run the SDP again. Repeating this procedure gives more and more sensitive tests. We will discuss an example later. In practice, this program can even decide separability if the O_i have already been measured, so it can be used to gain new insights into already performed experiments.

But before proceeding to the examples, let us note three more facts. First, in the formulation no dimension of the Hilbert space is fixed. Consequently, our approach is valid for any dimension, and combined with the methods of Ref. [19] it can be directly used to study multipartite entanglement in continuous-variable or hybrid systems [20]. For continuous variables, it can be employed complementary to the methods of Ref. [21].

Second, our approach can also be used to *quantify* genuine multipartite entanglement. If in Eq. (4) the trace normalization $\mathrm{Tr}(W) = 1$ is replaced by $0 \leq P_M \leq \mathbb{1}$ and $0 \leq Q_M \leq \mathbb{1}$, the negative witness expectation value is a multipartite entanglement monotone, since it obeys the following properties. (i) It vanishes on all biseparable states. (ii) It is convex. (iii) The quantity does not increase under protocols that consist of local operations of each party and classical communication between them. (iv) It is invariant under local basis changes. While most of these properties are straightforward to see—in particular, (iv) is implied by (iii)—the proof of property (iii) is more technical [22]. Note that, in the bipartite case, this monotone becomes the negativity [23].

Third, as mentioned before, there are other possible choices of supersets for the set of separable states, e.g., the set of states that have a symmetric extension on a larger Hilbert space [12,22].

Numerical examples.—We test the criterion of Eq. (4) for important pure three- and four-qubit states prepared in many experiments [24], by using the white noise tolerance as a figure of merit. It is given by the maximal amount p_{tol} of white noise for which the state $\varrho(p_{\mathrm{tol}}) = (1 - p_{\mathrm{tol}})|\psi\rangle\langle\psi| + p_{\mathrm{tol}}\mathbb{1}/2^n$ is still detected as entangled [25]. Table I shows the white noise tolerances of our criterion, compared with the most robust criteria so far.

Strikingly, the tolerances of the witnesses obtained by our SDP are significantly higher than previous ones. For the Greenberger-Horne-Zeilinger (GHZ) and the W state of three qubits and the GHZ and the linear cluster state of four qubits, we even obtain the best white noise tolerance possible, since we are able to show that for a larger amount of white noise the state becomes biseparable [22]. This shows that our criterion is indeed optimal for these cases.

TABLE I. White noise tolerances of the fully decomposable witnesses obtained by the SDP of Eq. (4) compared with the corresponding tolerances of the most robust criteria known so far. For the states marked by $\star$, we verified that adding more white noise than what is tolerated by Eq. (4) results in a biseparable state, so the values are optimal.

	White noise tolerances p_{tol}	
State	Fully decomposable	Before
$\lvert\mathrm{GHZ}_3\rangle^\star$	0.571	0.571 [7]
$\lvert\mathrm{GHZ}_4\rangle^\star$	0.533	0.533 [7]
$\lvert W_3\rangle^\star$	0.521	0.421 [7]
$\lvert W_4\rangle$	0.526	0.444 [7]
$\lvert\mathrm{Cl}_4\rangle^\star$	0.615	0.533 [26]
$\lvert D_{2,4}\rangle$	0.539	0.471 [27]
$\lvert\Psi_{S,4}\rangle$	0.553	0.317 [28]

To show that the criterion of Eq. (4) works well for a restricted set of observables, we consider the four-qubit Dicke state with two excitations $|D_{2,4}\rangle$ [24]. For this state, the SDP yields a witness W_D [22] that consists of the observables $\mathcal{O} = \{X^{\otimes 4}, Y^{\otimes 4}, Z^{\otimes 4}, X_1X_2Y_3Y_4, X_1X_2Z_3Z_4, Y_1Y_2Z_3Z_4\}$, their distinct permutations, and other observables that can be measured in the same run. For example, a local measurement of $X_1X_2X_3X_4$ yields knowledge of the expectation value of $X_1X_2\mathbb{1}_3X_4$. The SDP finds a witness consisting of $O_1 = X^{\otimes 4}$, $O_2 = Y^{\otimes 4}$, and observables obtained by replacing some Pauli operators by the identity. Already with these observables, the white noise tolerance is $p_{\mathrm{tol}}^{(2)} \approx 0.294\,95$. We can proceed in this way and use additional observables O_i from the set $\mathcal{O}$—including their permutations and observables obtained by replacing Pauli operators by $\mathbb{1}$—to produce strictly stronger witnesses $W_D^{(i)}$. Their white noise tolerances $p_{\mathrm{tol}}^{(i)}$ are $p_{\mathrm{tol}}^{(3)} \approx 0.383\,79$, $p_{\mathrm{tol}}^{(4)} \approx 0.383\,83$, $p_{\mathrm{tol}}^{(5)} \approx 0.452\,00$, and finally $p_{\mathrm{tol}}^{(6)} \approx 0.539\,14$ as in Table I, since $W_D = W_D^{(6)}$.

Third, we compute a lower bound on the volume of genuinely multipartite entangled states. We created samples of 10^4 random mixed three-qubit states uniformly distributed with respect to the Hilbert-Schmidt distance (or the Bures distance) and check whether they are genuinely multipartite entangled. 6.28% (Bures: 10.32%) were detected by fully decomposable and 0.44% (Bures: 1.06%) by fully PPT witnesses. As expected, fully PPT witnesses detect fewer states.

While the problem can still be tackled numerically for six or seven qubits, in recent experiments up to 14 ions have been coherently manipulated [29]. Therefore, we study analytical witnesses which can be generalized to an arbitrary number of qubits in the following.

Analytical results.—A fully decomposable witness for the four-qubit linear cluster state $|\mathrm{Cl}_4\rangle$ [24] that is obtained by the SDP of Eq. (4) is given by

$$W_{\mathrm{Cl4}} = \tfrac{1}{2}\mathbb{1} - |\mathrm{Cl}_4\rangle\langle\mathrm{Cl}_4| - \tfrac{1}{8}(\mathbb{1} - g_1)(\mathbb{1} - g_4), \qquad (5)$$

where $g_1 = Z_1Z_2\mathbb{1}_3\mathbb{1}_4$ and $g_4 = \mathbb{1}_1\mathbb{1}_2Z_3Z_4$ are two of the generators of the cluster state's so-called stabilizer group. This witness detects more states than the usual projector witness $W_{\mathrm{proj}} = \frac{1}{2}\mathbb{1} - |\mathrm{Cl}_n\rangle\langle\mathrm{Cl}_n|$, since W_{Cl4} is obtained from W_{proj} by subtracting a positive operator P_+. For n qubits, the generators are, after a local basis change, $g_1 = X_1Z_2$, $g_i = Z_{i-1}X_iZ_{i+1}$ for $1 < i < n$, and $g_n = Z_{n-1}X_n$. Then, the n-qubit linear cluster state is defined by $|\mathrm{Cl}_n\rangle\langle\mathrm{Cl}_n| = 2^{-n}\prod_{i=1}^{n}(\mathbb{1} + g_i)$. The construction of the four-qubit cluster state witness can be generalized to an arbitrary number of qubits [22]. For seven qubits, e.g., a witness is given by

$$W_{\mathrm{Cl}7} = \tfrac{1}{2}\mathbb{1} - |\mathrm{Cl}_7\rangle\langle\mathrm{Cl}_7| - \tfrac{1}{16}[(\mathbb{1} - g_1)(\mathbb{1} - g_4)(\mathbb{1} - g_7) + (\mathbb{1} + g_1)(\mathbb{1} - g_4)(\mathbb{1} - g_7) + (\mathbb{1} - g_1)(\mathbb{1} + g_4)(\mathbb{1} - g_7) + (\mathbb{1} - g_1)(\mathbb{1} - g_4)(\mathbb{1} + g_7)]. \tag{6}$$

For the case of n qubits, the white noise tolerance is

$$p_{\mathrm{tol}} = [1 - 2^{-n+1} + (k+1)2^{-k}]^{-1} \xrightarrow{n\to\infty} 1, \tag{7}$$

where $k = \lfloor \frac{n+2}{3} \rfloor$. This result is remarkable for several reasons. First, $W_{\mathrm{Cl}n}$ is the first example of a witness for genuine multipartite entanglement so far whose white noise tolerance converges to one for an increasing number of qubits. Thus, the volume of the largest ball inside the biseparable states around the totally mixed state approaches zero. A similar scaling behavior of the entanglement in the cluster state has been found in Ref. [30]. Note that, however, they considered full separability and not genuine multipartite entanglement. For full separability, this scaling behavior is not surprising, since it is known that the largest ball of fully separable states around the totally mixed states shrinks with an increasing particle number [31]. Moreover, the white noise tolerance of Eq. (7) corresponds to a required fidelity $F_{\mathrm{req}} \approx 1 - p_{\mathrm{tol}} \approx k2^{-k}$ for large n and therefore decreases exponentially fast with growing n. In contrast, the fidelity needed to detect entanglement by using W_{proj} equals one-half, independent of the particle number. Interestingly, this exponential improvement comes with very low experimental costs, since the additional term P_+ can be measured with only one experimental setting. Finally, note that our construction induces a similar construction for the 2D cluster state.

Discussion.—In this Letter, we presented an easily implementable criterion for genuine multipartite entanglement. We demonstrated its high robustness, connected it to entanglement measures, and provided powerful witnesses for an arbitrary number of qubits.

Because of its versatility, the criterion can be used to characterize the entanglement in various physical systems, e.g., in ground states of spin models undergoing a quantum phase transition. Moreover, it is a promising tool to study multipartite entanglement in continuous-variable systems. Finally, we believe that, as an easy-to-use scheme, it will be valuable for the analysis of experimental data that do not constitute a whole tomography.

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