Critical Gravity in Four Dimensions

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We study four-dimensional gravity theories that are rendered renormalizable by the inclusion of curvature-squared terms to the usual Einstein action with a cosmological constant. By choosing the parameters appropriately, the massive scalar mode can be eliminated and the massive spin-2 mode can become massless. This "critical" theory may be viewed as a four-dimensional analogue of chiral topologically massive gravity, or of critical "new massive gravity" with a cosmological constant, in three dimensions. We find that the on-shell energy for the remaining massless gravitons vanishes. There are also logarithmic spin-2 modes, which have positive energy. The mass and entropy of standard Schwarzschild-type black holes vanish. The critical theory might provide a consistent toy model for quantum gravity in four dimensions.

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Three-dimensional topologically massive gravity [1] has been studied extensively in recent times as a toy model for a quantum theory of gravity. In addition to the usual massless graviton (which carries no local degrees of freedom), there is in general a massive propagating spin-2 field. For generic values of the coupling constant μ^{-1} for the topological Chern-Simons term, the energy of massive spin-2 excitations is negative if one takes the Einstein-Hilbert term to have the conventional (positive) sign. On the other hand, the Banados-Teitelboim-Zanelli (BTZ) black hole has positive mass if the Einstein-Hilbert term has the conventional sign. Thus there is no choice of sign for the Einstein-Hilbert term which gives positivity for both the massive spin-2 excitations and the mass of the BTZ black-hole solution. It was, however, noted in Ref. [2] that if the Chern-Simons coupling is chosen so that $\mu\ell = 1$, where ℓ is the "radius" of the AdS_3 solution, then the massive spin-2 field becomes massless, and both the field excitations and the BTZ black-hole mass will be positive for the conventional choice of sign for the Einstein-Hilbert term. It was, furthermore, conjectured that the excitations in this "critical" theory are described by a consistent chiral two-dimensional boundary theory.

In this Letter, we address the question of whether any kind of analogous critical limit might arise in four-dimensional gravity. Even though such a limit would presumably not be expected to describe a realistic theory of four-dimensional gravity, it might nevertheless be of interest as another simplified "toy model," with the advantage in this case of being in four, rather than three, dimensions. Since one would hope that such a toy model would be renormalizable, the natural place to look is in the class of four-dimensional gravity theories with curvature-squared modifications, which were first studied in detail in Refs. [3,4]. Since the Gauss-Bonnet invariant does not contribute to the equations of motion in four dimensions, we just need to consider the action

$$I = \frac{1}{2\kappa^2}\int \sqrt{-g}d^4x(R - 2\Lambda + \alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2). \quad (1)$$

As was discussed in [3,4] (for $\Lambda = 0$), this theory is renormalizable, and it describes in general a massless spin-2 graviton, a massive spin-2 field, and a massive scalar. The energies of excitations of the massive spin-2 field are negative, while those of the massless graviton are, as usual, positive. Thus although the theory is renormalizable, it suffers from having ghosts. The massive spin-0 is absent in the special case $\alpha = -3\beta$, while the massive spin-2 is absent if instead $\alpha = 0$.

Following the general strategy of Ref. [2], we shall seek a limit in which the massive spin-2 field becomes massless. The presence of the cosmological constant in the action (1) is essential for this step. We shall also choose the parameters so that the massive scalar is absent. The energies of excitations of the remaining massless graviton then vanish. The fourth-order graviton operator also admits logarithmic modes, and we find that these have positive energy. We then investigate the mass of black holes in the critical theory. We find that the effect of the curvature-squared terms is to modify the mass formula such that Schwarzschild-anti–de Sitter (AdS) black holes are massless. (It is possible that there may exist more general black-hole solutions of the fourth-order equations, even static ones, which might have positive mass.) In fact, three-dimensional "new massive gravity" [5] with a cosmological constant exhibits similar features at its critical point, of having zero-energy excitations for the massless gravitons [6] and zero mass for the BTZ black hole [7].

It is, of course, rather unusual to have a theory of gravity in which black holes are massless. One may have to accept this as the price to be paid for having a four-dimensional theory of gravity, without ghosts, that has the possibility of being renormalizable. Thus, although the theory cannot claim to be in any way phenomenologically realistic, it may provide a useful simplified arena for studying some aspects of a potentially renormalizable theory of massless spin-2 fields in four dimensions.

The equations of motion that follow from the action (1) are $\mathcal{G}_{\mu\nu} + E_{\mu\nu} = 0$, where

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{2}$$

$$\begin{aligned} E_{\mu\nu} = {} & 2\alpha(R_{\mu\rho}R^{\rho}_{\nu} - \tfrac{1}{4}R^{\rho\sigma}R_{\rho\sigma}g_{\mu\nu}) + 2\beta R(R_{\mu\nu} - \tfrac{1}{4}Rg_{\mu\nu}) \\ & + \alpha(\Box R_{\mu\nu} + \tfrac{1}{2}\Box R g_{\mu\nu} - 2\nabla_\rho\nabla_{(\mu}R_{\nu)}{}^{\rho}) \\ & + 2\beta(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R). \end{aligned} \tag{3}$$

In what follows, we shall need to consider the linearization of these equations around a background solution of the equations of motion. We shall take the background to be four-dimensional anti–de Sitter spacetime, for which, following from the equations of motion,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \qquad R_{\mu\nu\rho\sigma} = \frac{\Lambda}{3}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{4}$$

Note that in four dimensions, unlike in higher dimensions, the inclusion of the explicit cosmological constant in (1) is essential in order to have an AdS solution. This is because $E_{\mu\nu}$ vanishes in any Einstein space background in four dimensions.

Writing the varied metric as $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, and so $\delta g_{\mu\nu} = h_{\mu\nu}$, we find to first order in variations that

$$\begin{aligned} \delta(\mathcal{G}_{\mu\nu} + E_{\mu\nu}) = {} & [1 + 2\Lambda(\alpha + 4\beta)]\mathcal{G}^L_{\mu\nu} \\ & + \alpha\left[\left(\Box - \frac{2\Lambda}{3}\right)\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{3}R^L g_{\mu\nu}\right] \\ & + (\alpha + 2\beta)[-\nabla_\mu\nabla_\nu + g_{\mu\nu}\Box + \Lambda g_{\mu\nu}]R^L, \end{aligned} \tag{5}$$

where $\mathcal{G}^L_{\mu\nu}$ and R^L are the linearized variations of $\mathcal{G}_{\mu\nu}$ and R:

$$\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} - \tfrac{1}{2}R^L g_{\mu\nu} - \Lambda h_{\mu\nu}, \tag{6}$$

$$R^L_{\mu\nu} = \nabla^\lambda\nabla_{(\mu}h_{\nu)\lambda} - \tfrac{1}{2}\Box h_{\mu\nu} - \tfrac{1}{2}\nabla_\mu\nabla_\nu h, \tag{7}$$

$$R^L = \nabla^\mu\nabla^\nu h_{\mu\nu} - \Box h - \Lambda h. \tag{8}$$

(We have also defined $R^L_{\mu\nu}$, the linearization of $R_{\mu\nu}$, and introduced $h = g^{\mu\nu}h_{\mu\nu}$.)

For our purposes, it will be convenient to use general coordinate invariance to impose the gauge condition

$$\nabla^\mu h_{\mu\nu} = \nabla_\nu h. \tag{9}$$

Substituting this into (6)–(8), we find

$$\begin{aligned} \mathcal{G}^L_{\mu\nu} &= -\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{2}\nabla_\mu\nabla_\nu h + \frac{\Lambda}{3}h_{\mu\nu} + \frac{\Lambda}{6}h, \\ R^L &= -\Lambda h. \end{aligned} \tag{10}$$

We can substitute these expressions into Eq. (5). Taking the trace, we find

$$0 = g^{\mu\nu}\delta(\mathcal{G}_{\mu\nu} + E_{\mu\nu}) = \Lambda[h - 2(\alpha + 3\beta)\Box h]. \tag{11}$$

We see that h describes a propagating massive scalar mode, except in the special case that

$$\alpha = -3\beta, \tag{12}$$

in which case the equations of motion imply that $h = 0$. It is this case, where (12) holds, that we shall focus on in our subsequent discussion [8]. Note that modulo the Gauss-Bonnet combination, which does not contribute to the equations of motion in four dimensions, the curvature-squared terms with $\alpha = -3\beta$ can be written as $\frac{1}{2}\alpha C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

Having imposed (12), implying that $h = 0$, the variation of the field equations gives

$$0 = \frac{3\beta}{2}\left(\Box - \frac{2\Lambda}{3}\right)\left(\Box - \frac{4\Lambda}{3} - \frac{1}{3\beta}\right)h_{\mu\nu}, \tag{13}$$

where $h_{\mu\nu}$ is in the transverse traceless gauge

$$\nabla^\mu h_{\mu\nu} = 0, \qquad g^{\mu\nu}h_{\mu\nu} = 0. \tag{14}$$

The fourth-order Eq. (13) describes a massless graviton, satisfying

$$\left(\Box - \frac{2\Lambda}{3}\right)h^{(m)}_{\mu\nu} = 0, \tag{15}$$

and a massive spin-2 field, satisfying

$$\left(\Box - \frac{4\Lambda}{3} - \frac{1}{3\beta}\right)h^{(M)}_{\mu\nu} = 0. \tag{16}$$

The criterion of stability for spin-2 modes satisfying $(\Box - 2\Lambda/3 - M^2)h_{\mu\nu} = 0$ in the AdS_4 background requires that $M^2 \geq 0$ (see, for example, [9]), and so, since Λ is negative, we must have

$$0 < \beta \leq \left(-\frac{1}{2\Lambda}\right). \tag{17}$$

(In particular, β must be positive.) We shall choose the critical value

$$\beta = -\frac{1}{2\Lambda}. \tag{18}$$

By imposing (12) in order to eliminate the massive scalar mode, and additionally (18) in order to make the massive spin-2 mode become massless, we have arrived at a four-dimensional theory describing only massless gravitons. We may now examine the energy of the excitations of the graviton modes in the AdS_4 background,

$$ds_4^2 = \frac{3}{(-\Lambda)}[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2]. \quad (19)$$

A procedure for doing this has been described in Ref. [2], based upon the construction of the Hamiltonian for the graviton field. Leaving β as yet unrestricted, we may write down the quadratic action whose variation yields the equations of motion (13):

$$\begin{aligned} I_2 &= -\frac{1}{2\kappa^2}\int\sqrt{-g}d^4x h^{\mu\nu}(\delta\mathcal{G}_{\mu\nu} + \delta E_{\mu\nu}) \\ &= -\frac{1}{2\kappa^2}\int\sqrt{-g}d^4x\Big[\frac{1}{2}(1+6\beta\Lambda)(\nabla^\lambda h^{\mu\nu})(\nabla_\lambda h_{\mu\nu}) \\ &\quad + \frac{3}{2}\beta(\Box h^{\mu\nu})(\Box h_{\mu\nu}) + \frac{\Lambda}{3}(1+4\beta\Lambda)h^{\mu\nu}h_{\mu\nu}\Big]. \quad (20) \end{aligned}$$

Using the method of Ostrogradsky for Lagrangians written in terms of second, as well as first, time derivatives, we define the conjugate "momenta"

$$\begin{aligned} \pi^{(1)\mu\nu} &= \frac{\delta L_2}{\delta\dot{h}_{\mu\nu}} - \nabla_0\Big(\frac{\delta L_2}{\delta(d(\nabla_0 h_{\mu\nu})/dt)}\Big) \\ &= -\frac{1}{2\kappa^2}\sqrt{-g}\nabla^0((1+6\beta\Lambda)h^{\mu\nu} - 3\beta\Box h^{\mu\nu}), \\ \pi^{(2)\mu\nu} &= \frac{\delta L_2}{\delta(d(\nabla_0 h_{\mu\nu})/dt)} = -\frac{3\beta}{2\kappa^2}\sqrt{-g}g^{00}\Box h^{\mu\nu}. \quad (21) \end{aligned}$$

Since the Lagrangian is time-independent, the Hamiltonian is equal to its time average, and writing it in this way is advantageous because we can then integrate time derivatives by parts. Thus we obtain the Hamiltonian

$$\begin{aligned} H &= T^{-1}\Big(\int d^4x\Big[\pi^{(1)\mu\nu}\dot{h}_{\mu\nu} + \pi^{(2)\mu\nu}\frac{\partial(\nabla_0 h_{\mu\nu})}{\partial t}\Big] - I_2\Big) \\ &= \frac{1}{2\kappa^2 T}\int\sqrt{-g}d^4x\Big[-(1+6\beta\Lambda)\nabla^0 h^{\mu\nu}\dot{h}_{\mu\nu} \\ &\quad + 6\beta\Big(\frac{\partial}{\partial t}(\Box h^{\mu\nu})\Big)(\nabla^0 h_{\mu\nu})\Big] - \frac{1}{T}I_2, \quad (22) \end{aligned}$$

where all time integrations are taken over the interval T.

Evaluating this for the massless mode [satisfying (15)] and for the massive mode [satisfying (16)], we therefore obtain the on-shell energies

$$E_{\mathrm{massless}} = -\frac{1}{2\kappa^2 T}(1+2\beta\Lambda)\int\sqrt{-g}d^4x\nabla^0 h^{\mu\nu}_{(m)}\dot{h}^{(m)}_{\mu\nu}, \quad (23)$$

$$E_{\mathrm{massive}} = \frac{1}{2\kappa^2 T}(1+2\beta\Lambda)\int\sqrt{-g}d^4x\nabla^0 h^{\mu\nu}_{(M)}\dot{h}^{(M)}_{\mu\nu}. \quad (24)$$

Evidently, since the graviton modes in pure Einstein gravity with $\beta = 0$ are known to have positive energy, the integral itself in (23) must be negative. The integral in (24) is therefore also expected to be negative (at least when β is chosen so that the mass of the mode is small, and probably in all cases), and so we see that the massive excitations in AdS, with positive β, will have negative energy. Imposing now our criticality condition (18), we see that the energies (23) and (24) become equal and vanish. This is analogous to the critical situation [6] of new massive gravity [5] with a cosmological constant.

There are also logarithmic modes at the critical point, which are annihilated by the full fourth-order operator $(\Box + \frac{2}{3}\Lambda)^2$ but not by $(\Box + \frac{2}{3}\Lambda)$ alone. (Analogous modes in three-dimensional chiral gravity were obtained in Ref. [10].) Such logarithmic modes have been constructed recently in Ref. [11]; they can be written in the form $h^{\log}_{\mu\nu} = f(t,\rho)h_{\mu\nu}$, where $h_{\mu\nu}$ are standard spin-2 massless modes and $f(t,\rho) = 2it + \mathrm{logsinh}2\rho - \mathrm{logtanh}\rho$. (See also [12].) We have explicitly evaluated the expression (22) at the critical point for these modes and found, by numerical integration, that they have a finite and strictly positive energy [13].

We now investigate the mass of black-hole solutions. (All solutions of the $\alpha = \beta = 0$ theory are also solutions of the full theory.) This can be done by using the procedure of Abbott and Deser [14], by writing the black-hole metric in the form $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the metric on AdS, and interpreting the linearized variation of the field equation, given in our case by (5), as an effective gravitational energy-momentum tensor $T_{\mu\nu}$ for the black-hole field. One then writes the conserved current $J^\mu = T^{\mu\nu}\xi_\mu$, where ξ^μ is a Killing vector that is timelike at infinity, as the divergence of a 2-form $\mathcal{F}_{\mu\nu}$, i.e., $J^\mu = \nabla_\nu\mathcal{F}^{\mu\nu}$. From this, one obtains the Abbott-Deser mass

$$E = \frac{1}{2\kappa^2}\int_{S_\infty} dS_i\mathcal{F}^{0i}, \quad (25)$$

as an integral over the sphere at infinity. The relevant contributions to $\mathcal{F}^{\mu\nu}$ associated with the various terms in (5) have been calculated in Ref. [15]. By defining

$$\begin{aligned} \mathcal{F}^{\mu\nu}_{(0)} &= \xi_\alpha\nabla^{[\mu}h^{\nu]\alpha} + \xi^{[\mu}\nabla^{\nu]}h + h^{\alpha[\mu}\nabla^{\nu]}\xi_\alpha \\ &\quad - \xi^{[\mu}\nabla_\alpha h^{\nu]\alpha} + \tfrac{1}{2}h\nabla^\mu\xi^\nu, \\ \mathcal{F}^{\mu\nu}_{(1)} &= 2\xi^{[\mu}\nabla^{\nu]}R^L + R^L\nabla^\mu\xi^\nu, \\ \mathcal{F}^{\mu\nu}_{(2)} &= -2\xi_\alpha\nabla^{[\mu}\mathcal{G}_L^{\nu]\alpha} - 2\mathcal{G}_L^{\alpha[\mu}\nabla^{\nu]}\xi_\alpha, \end{aligned} \quad (26)$$

it follows that

$$\begin{aligned} \nabla_\nu\mathcal{F}^{\mu\nu}_{(0)} &= \mathcal{G}_L^{\mu\nu}\xi_\nu, \\ \nabla_\nu\mathcal{F}^{\mu\nu}_{(1)} &= [(-\nabla_\mu\nabla_\nu + g^{\mu\nu}\Box + \Lambda g^{\mu\nu})R^L]\xi_\nu, \\ \nabla_\nu\mathcal{F}^{\mu\nu}_{(2)} &= \Big[\Big(\Box - \frac{2\Lambda}{3}\Big)\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{3}R^L g^{\mu\nu}\Big]\xi_\nu. \end{aligned} \quad (27)$$

These are precisely the three structures in (5), after contracting $\delta(\mathcal{G}_{\mu\nu} + E_{\mu\nu})$ with ξ^ν, so $\mathcal{F}^{\mu\nu} = \sum_i\mathcal{F}^{\mu\nu}_{(i)}$.

Carrying out this procedure for the Schwarzschild-AdS black hole, one finds that only the term in (25) coming from $\mathcal{F}^{\mu\nu}_{(0)}$ gives a nonvanishing contribution, and therefore the Abbott-Deser mass is [16]

$$M = m[1 + 2\Lambda(\alpha + 4\beta)] = m(1 + 2\beta\Lambda), \qquad (28)$$

where m is the usual mass parameter of the solution. Thus the black-hole mass is non-negative for β in the range (17). For our critical condition (18) that makes the massive spin-2 mode become massless, we see that (28) becomes $M = 0$, and so the Schwarzschild-AdS black hole has zero mass. (We expect the same to be true of Kerr-AdS black holes.) Similar zero-energy results have previously been obtained in the context of a scale-invariant gravity theory with a pure Weyl-squared action [17] and in three-dimensional critical new massive gravity [7]. (See also [18].)

Wald's formula $S = -2\pi \int \sqrt{h} d^2x \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}(\partial \mathcal{L}/\partial R_{\alpha\beta\gamma\delta})$ [19] for the entropy gives $S = \pi r_+^2[1 + 2\Lambda(\alpha + 4\beta)]$ for the Schwarzschild-AdS black hole, which is consistent with the first law of thermodynamics $dM = TdS$ for all α and β, including the critical case $\alpha = -3\beta = 3/(2\Lambda)$ where both M and S vanish.

It was shown in Ref. [3] that four-dimensional Einstein gravity with curvature-squared terms added is in general renormalizable. The case $\alpha = -3\beta$ was, however, excluded, on the grounds that the scalar mode would then have a propagator with only $1/k^2$ falloff rather than $1/k^4$. A key new feature in our discussion is the inclusion of a cosmological constant in the theory, leading, as we saw from Eq. (11), to the entire elimination of the scalar mode when $\alpha = -3\beta$. Whether the critical theory we have constructed might in fact be renormalizable now depends on whether $\alpha = -3\beta$ and $\beta = -1/(2\Lambda)$ are stable under the renormalization group flow. A renormalization group analysis of the stability of the chiral point in three-dimensional topologically massive gravity suggests that, in that model, the chiral point is not stable [20]. However, differences between that theory and our critical gravity, together with possible subtleties of scheme dependence, leave this as an open question at this point.

In this Letter, we have studied a four-dimensional theory of gravity with a cosmological constant, possibly rendered renormalizable by including curvature-squared terms in the action. By choosing parameters appropriately, we eliminated the massive scalar mode that is generically present, and we also arranged for the massive spin-2 mode to become massless. The resulting critical theory could be viewed as a four-dimensional analogue of the chiral three-dimensional topologically massive gravity theory studied in Ref. [2]. Although the massless spin-2 modes have zero energy, the fact that logarithmic modes of the fourth-order graviton operator have positive energy shows that the critical theory is not entirely trivial. The standard Schwarzschild-AdS black hole has zero mass and (consistently) zero entropy; there could in principle exist more general black-hole solutions that are not also solutions of pure Einstein gravity. It would be of interest to construct these and to see if they have nonzero mass.

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Note added in proof.—Recent results in Refs. [21,22] have shown that a general linear combination of the regular and logarithmic spin-2 modes has an indefinite norm. Thus, imposing unitarity would require suppressing the logarithmic modes.

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