Entanglement in the Symmetric Sector of n Qubits

P. Ribeiro*
CFIF, Instituto Superior Técnico, Universidade Técnica de Lisboa, Avenida Rovisco Pais, 1049-001 Lisboa, Portugal

R. Mosseri†
Laboratoire de Physique Théorique de la Matière Condensée, CNRS UMR 7600, Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris Cedex 05, France
(Received 14 January 2011; published 3 May 2011)

We discuss the entanglement properties of symmetric states of n qubits. The Majorana representation maps a generic such state into a system of n points on a sphere. Entanglement invariants, either under local unitaries (LU) or stochastic local operations and classical communication (SLOCC), can then be addressed in terms of the relative positions of the Majorana points. In the LU case, an overcomplete set of invariants can be built from the inner product of the radial vectors pointing to these points; this is detailed for the well-documented three-qubits case. In the SLOCC case, a cross ratio of related Möbius transformations are shown to play a central role, exemplified here for four qubits. Finally, as a side result, we also analyze the manifold of maximally entangled 3 qubit state, both in the symmetric and generic case.

DOI: 10.1103/PhysRevLett.106.180502 PACS numbers: 03.67.Mn, 03.65.Ud

The potential power of quantum information, either for cryptography and computation purposes, is largely based on the subtle concept of quantum entanglement [1]. In a system composed of n two-level entities (qubits), a generic state is entangled; i.e., it cannot be written as a separable product of states belonging to each constitutive part. While it is rather easy to characterize entanglement for a 2-qubit system, the task of quantifying the amount of entanglement carried by the total system is very difficult, for increasing n.

Several entanglement measures have, nevertheless, been proposed (see [2,3] for comprehensive reviews), and their behavior under state transformation studied. Important cases are given by those quantities which remain invariant under (stochastic) local operations and classical communication, noted (S)LOCC [4,5]. Stated as operations performed in the multiquibit Hilbert space $\mathcal{H}$, the latter read $\otimes_i M_i$, called local unitaries (LU) for LOCC (with M_i a unitary matrix), and invertible local operations (ILO) for SLOCC (M_i a matrix with nonvanishing determinant).

One aims to find a complete set of such invariants that parametrizes the orbit space $\mathcal{H}/\otimes_i M_i$. Physically this means that states can only be obtained from each other with a local transformation (LU or ILO) if they share the same set of invariants. In the LOCC case, LU invariants can in principal be written as polynomial functions of the state components [6–8]. However their number proliferates with n, and finding explicit expressions becomes challenging; moreover their physical relevance is not necessarily obvious. Upon enlarging the set of operations that can be performed locally, like passing from LU to ILO, the number of entanglement classes can generally be reduced.

In this Letter we consider symmetric n-qubit states, and analyze their entanglement properties under LOCC and SLOCC. Such states have been the subject of several recent studies [9–16], with even some experimental [17] realizations or proposals [18]. In that case most of the relevant bipartite entanglement criteria were shown to coincide [19] and generic entanglement measures usually simplify.

We use the Majorana representation [20], which characterizes such a state as a collection of n points on a sphere, and derive the entanglement invariants in terms of the points arrangement. In the LOCC case, invariants can indeed be built from the inner product of the radial vectors pointing to these points; we explicitly derive the well-known 6 LU invariants for three qubits. In the SLOCC case, we show how sets of cross-ratio invariants under ILO related Möbius transformations play a central and clarifying role, and relate to a recently proposed classification of entanglement classes [21]. For four qubits, the most generic SLOCC invariant is simply related to the Klein modular invariant [22]. Finally, and as a side result, we also precise the manifold of maximally entangled 3 qubits Greenberger-Horne-Zeilinger (GHZ)-like states, both in the symmetric and the generic cases.

Majorana representation in the symmetric sector.—The n-qubits Hilbert space decomposes into subspaces of constant total spin $S^2 = \mathbf{S}.\mathbf{S}$ (where $S = \frac{1}{2}\sum_{i=1}^{n}\boldsymbol{\sigma}_i$). The subspace of maximal spin, $S^2 = s(s+1)$ with $n = 2s$, which appears once in this decomposition, corresponds to the fully symmetric sector, spanned by the Dicke basis ($S_z|s,m\rangle = m|s,m\rangle$). Using spin coherent states $|\alpha\rangle = e^{\alpha S_+}|s, m = -s\rangle$, where $S_\pm = S_x \pm iS_y$, any symmetric state $|\Psi\rangle$ can be represented by its Majorana polynomial

$$\Psi(\alpha) = \sum_{m=-s}^{s}\sqrt{\frac{(2s)!}{(s-m)!(s+m)!}}\langle s,m|\Psi\rangle\alpha^{m+s}. \qquad (1)$$

Up to a global unphysical factor, this state is therefore fully characterized by the set $\{\alpha_i\}$, made of the n complex zeros of $\Psi(\alpha)$, suitably completed by points at infinity whenever $\langle s, s|\Psi\rangle$ vanishes: $\Psi(\alpha) \propto \prod_{i=1}^{2s}(\alpha - \alpha_i)$. A nice geometrical representation of $|\Psi\rangle$, by n points on the unit sphere, is obtained by an inverse stereographic map of $\{\alpha_i\} \to \{\mathbf{v}_i\}$. The Majorana high spin spherical representation generalizes (although published earlier) the spin $1/2$ Bloch sphere; it recently proved quite useful in the context of collective spin models [23].

Symmetric LU Invariants (SLUI).—A generic local (separable) unitary transformation acting on a set of n qubits can be written, up to a unphysical phase, in the form $U = \otimes_i e^{(i/2)\mathbf{h}_i\cdot\sigma_i}$, with a collection of vectors $\mathbf{h}_{i=1,\ldots,n} \in \mathbb{R}^3$. In the symmetric sector, we restrict to identical $\mathbf{h}_i$, leading to the simpler form $U_s = e^{i\mathbf{h}\cdot\mathbf{S}}$. This implies that, in the symmetric sector, the only states that are LU equivalent correspond to sets of (unordered) Majorana zeros which can be transformed into each other by a global rotation of their representative vectors $\mathbf{v}_i \to \tilde{\mathbf{v}}_i = R\cdot\mathbf{v}_i$, with R in SO(3). Moreover, one also expects equivalent entanglement measures for states that are related by an (antiunitary) time reversal operation $T = \otimes_{j=1}^n (i\sigma_y)\mathcal{K}$, where $\mathcal{K}$ is the complex conjugate operator in the computational basis $\mathcal{K}(\sum_{ijk=0,1} t_{i,j,k}|i,j,k\rangle) = (\sum_{ijk=0,1} \bar{t}_{i,j,k}|i,j,k\rangle)$ and $T^2 = (-1)^n$. Geometrically, this corresponds to an inversion $\mathbf{v}_i \to \tilde{\mathbf{v}}_i = -\mathbf{v}_i$.

An overcomplete set of SLUI is obtained from symmetrized products of the innerproducts $v_{ij} = \mathbf{v}_i\cdot\mathbf{v}_j$, as, for instance, with the c_k coefficients of x^k in the symmetrized product $\prod_{ij}(x - v_{ij}) = \sum c_k x^k$. It is instructive to relate them to the standard invariants for two and three qubits. We make use of density matrices $\rho = |\Psi\rangle\langle\Psi|$ and eventually use their partial trace, with indices in ρ indicating those parts which have not been traced out.

The two-qubits case.—For 2 qubits, there is one entanglement invariant (if we disregard the trivial invariant $\mathrm{Tr}[\rho] = 1$ for a normed state), which we express here with the single inner product v_{12}. It can be given as the (equal) radius r_i of the partial Bloch sphere when tracing out one of the 2 subsystems. From $r_i^2 = 2\,\mathrm{Tr}[\rho_i^2] - 1$, one gets $r_i = \frac{8(v_{12}+1)}{(v_{12}+3)^2}$. Another most used form is the concurrence [24] running from zero for a separable state to unity for a maximally entangled EPR state. In the symmetric sector, it reads $C = \frac{4}{v_{12}+3} - 1$. Separable symmetric states correspond to the case with the two identical Majorana points, while symmetric EPR corresponds to pairs of antipodal points ($v_{12} = -1$). The latter set is then given by the sphere S^2 with opposite points identified, the projective plane RP^2. Note that a simple but careful analysis, not reproduced here, allows us to extend the EPR case to the full Hilbert space (not only to the symmetric sector), and recover the well-known $RP^3[\equiv \mathrm{SO}(3)]$ EPR manifold [25].

The three-qubits case.—A complete set of six independent LU invariant polynomials is known [26,27]. For a generic 3-qubit state, $I_{i=2,3,4} = 2\,\mathrm{Tr}[\rho_{i-1}^2] - 1$, $I_5 = \mathrm{Tr}[3(\rho_1\otimes\rho_2)\cdot\rho_{12}] - \mathrm{Tr}[\rho_1^3] - \mathrm{Tr}[\rho_2^3]$, $I_6 = \tau_3$. Again, $I_1 = \mathrm{Tr}[\rho] = 1$ for a normed state. $I_{2,3,4}$ are related to the radius of the (partial) Bloch balls of qubits (1, 2, 3), respectively, once the other two are traced out. I_5 is the Kempe invariant [26] and I_6 the 3-tangle, which takes the form of a hyperdeterminant [27]. Note that $I_{1,\ldots,6}$ are also invariant under a time reversal transformation. Restricted to the symmetric sector, these invariants explicitly read, with $c_0 = -v_{12}v_{13}v_{23}$, $c_1 = v_{12}v_{13} + v_{12}v_{23+}v_{13}v_{23}$, and $c_2 = -(v_{12} + v_{13} + v_{23})$:

$$
\begin{aligned}
I_{2,3,4} &= \frac{-6c_0 + 18c_1 + (c_2 - 60)c_2 + 75}{9(c_2 - 3)^2},\\
I_5 &= \frac{1}{18(c_2 - 3)^3}[-9c_0(c_2 - 9) - 459\\
&\quad + 27c_1(c_2 - 5) + (c_2 - 24)c_2(4c_2 - 21)],\\
I_6 &= \frac{2(c_0 + c_1 + c_2 + 1)}{3(c_2 - 3)^2}.
\end{aligned}
\tag{2}
$$

Using $\theta_{i,j} = \arccos v_{i,j}$ as coordinate axes, and recalling that the set of Majorana points is not ordered, we can display the symmetric sector entanglement types inside the tetrahedron ($OABC$) shown in Fig. 1. Analyzing the subgroups of SO(3) that leave each representative state invariant one can characterize the manifold corresponding to each entanglement family (see Table I).

Toward a determination of the unit 3-tangle manifold.—Symmetric GHZ states (with unit 3-tangle $I_6 = 1$) correspond to the three Majorana points forming an equilateral triangle on an equatorial plane. The set of equatorial planes is the projective plane RP^2. Adding the triangles global rotation modulo $2\pi/3$, the set of symmetric unit 3-tangle states inherits the geometry $\mathrm{SO}(3)/Z_3$.

Using the above defined time reversal operator T, we consider the operator $Y(\theta) = (\cos\theta + \sin\theta T)$, whose

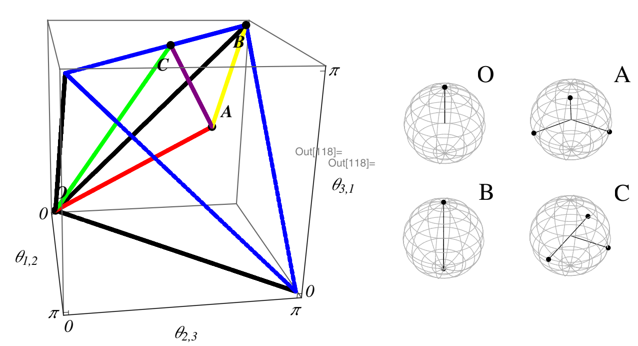

FIG. 1 (color online). Entanglement types, for symmetric 3-qubits. Point O corresponds to separable states (with coinciding 3 Majorana points), B and A to W and GHZ states, respectively. States corresponding to point C can be brought to the form $\frac{1}{\sqrt{2}}(|s = \frac{3}{2}, m = -\frac{1}{2}\rangle + |s = \frac{3}{2}, m = \frac{1}{2}\rangle)$ by a suitable LU.

TABLE I. Manifold of the particular points O, A, B, and C of Fig. 1.

States	Manifold	I_2	I_5	I_6
O	S^2	1	1	0
A	$SO(3)Z_3$	0	1/4	1[b]
B	S^2	1/9	2/9[a]	0
C	SO(3)	4/9	17/36	1/3

[a]2/9 is the minimum of I_5 within the class of symmetric states arising only for type B states.
[b]Maximal 3-tangle states.

inverse is $Y(-\theta)$ (since $T^2 = -1$ for n odd); $Y(\theta)$ is left unchanged under conjugation with a LU. Applying $Y(\theta)$ onto a separable 3-qubit state, one gets interesting entangled states. Starting from a symmetric separable state, one proves that any symmetric GHZ state can be obtained as $|\Psi\rangle = Y(\frac{\pi}{4})|u\rangle|u\rangle|u\rangle$. More generically, $Y(\frac{\pi}{4})$ maps a nonsymmetric separable state $|u_1\rangle|u_2\rangle|u_3\rangle$ onto a (nonsymmetric) GHZ state, as can be verified by a direct check. One can show that these GHZ states form the manifold $\mathcal{M} = S^2 \times S^2 \times SO(3)/Z_3$. In the case (yet unproved, but numerically plausible) that any generic unit 3-tangle GHZ can be sent to the symmetric space by a LU, this would prove that the full GHZ manifold is indeed $\mathcal{M}$. Note $\mathcal{M}$ differs by a factor Z_3 from that given in [28].

Symmetric states SLOCC invariants.—A nice description of SLOCC invariant families was recently proposed for symmetric n-qubits states [21,29], which focuses on the number of different roots α_i and their degeneracy. This allows a full classification for $n = 2$ or 3 but, as stressed by the authors, leaves continuous families of additional parameters for larger n. Our aim here is to provide a closer look to this question, by mapping this problem to the classification of invariants of Möbius transformations. Indeed, an ILO A that leaves the symmetric sector invariant can also be parametrized as U_s, but now with $\mathbf{h}$ being complex instead of real. Upon simple manipulations, one parametrizes this transformation as $A = e^{ih((1/\beta_1+\beta_2)S_+ + S_z - (\beta_1\beta_2/\beta_1+\beta_2)S_-)}$, where β_1, β_2, $h \in \mathbb{C}$. The action of this operator on a generic state in the coherent state basis is given by [30]

$$A\Psi(\alpha) = \left[\frac{\gamma^{-1}(\alpha - \beta_1) - \gamma(\alpha - \beta_2)}{(\beta_1 - \beta_2)}\right]^{2s} \times \Psi\left(\frac{\gamma^{-1}\beta_2(\alpha - \beta_1) - \gamma\beta_1(\alpha - \beta_2)}{\gamma^{-1}(\alpha - \beta_1) - \gamma(\alpha - \beta_2)}\right), \quad (3)$$

where $\gamma = e^{i(h/2)[(\beta_1 - \beta_2)/(\beta_1 + \beta_2)]}$. Note that this transformation lets the wave function invariant for $\alpha = \beta_1$ and $\alpha = \beta_2$. The roots α_i of the polynomial $\Psi(\alpha)$ transform according to the following Möbius transformation (MT):

$$\alpha_i \to \alpha_i' = \frac{(\beta_2\gamma - \beta_1\gamma^{-1})\alpha_i + \beta_1\beta_2(\gamma^{-1} - \gamma)}{(\gamma - \gamma^{-1})\alpha_i + \gamma^{-1}\beta_2 - \gamma\beta_1}. \quad (4)$$

Unitary transformations are recovered whenever $\beta_1 = -\bar{\beta}_2^{-1}$ and $h \in \mathbb{R}$, corresponding to the subclass of elliptic MT. This mapping from ILO to MT is particularly interesting when looking to invariant quantities. Indeed, the latter are well known to preserve the "cross-ratio" of four (here complex) numbers:

$$(\alpha_i, \alpha_j; \alpha_k, \alpha_l) = \frac{(\alpha_i - \alpha_k)(\alpha_j - \alpha_l)}{(\alpha_j - \alpha_k)(\alpha_i - \alpha_l)}, \quad (5)$$

which therefore form the natural building blocks for SLOCC invariants. Note that permuting the roots α in the cross ratio $(\alpha_i, \alpha_j; \alpha_k, \alpha_l) = \lambda$ leads generically to the following six different values for the cross ratio out of the 24 permutations: $\{\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\}$, belonging to distinct regions in the complex plane (Fig. 2).

As discussed in [21], for n qubits, the symmetric SLOCC classes are parametrized by $n - 3$ continuous parameters. In terms of MT, this is nothing but the known property that a unique MT relate two sets of three distinct complex numbers, and that transformations involving n complex numbers are parametrized by $n - 3$ cross ratios. This immediately recovers the result that, for $n = 3$, there are 3 SLOCC classes in the symmetric sector, labeled by the points O, B, and A in Fig. 1: separable states (point O), with the three roots α_i identical, W states (point B) with two roots identical, and the remaining (generic) states that can be mapped under SLOCC to the GHZ state (point A).

A complete set of SLOCC invariants (for any n) can be obtained by choosing 3 roots α_i ($i = 1, 2, 3$) in order to define the function $\lambda(z) = \frac{(z-\alpha_1)(\alpha_2-\alpha_3)}{(z-\alpha_3)(\alpha_2-\alpha_1)}$. The $n - 3$ complex values $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_{n-3}\}$, where $\lambda_{j-3} = \lambda(\alpha_j)$

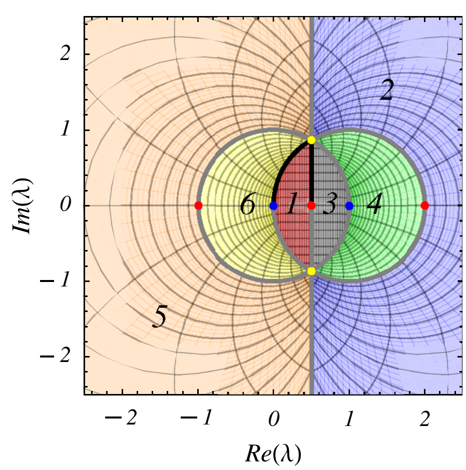

FIG. 2 (color online). Symmetries of the cross product. For a given set of four complex numbers, the 6 permutation related cross ratios belong to separate regions $\mathcal{D}_I$ labeled here from 1 to 6. The boundaries of the regions carry more symmetries, so one should, for example, only consider the black lines for region $\mathcal{D}_1$. States associated with invariant on the boundary set, like the colored ones, are expected to display particular properties.

for each $\alpha_{j>3}$, form the SLOCC invariants. Since the ordering of the n roots is arbitrary, there are in general $n!$ such sets: under a permutation Π the cross ratios transform as $\boldsymbol{\lambda} \rightarrow \boldsymbol{\lambda}'(\Pi)$ where each $\lambda'_j(\Pi)$ is a rational function of the λ_j's.

For $n = 4$, we noted the reduction to six independent transformations; the requirement that $\lambda = \lambda(\alpha_4) \in \mathcal{D}_1$ fixes then a unique value of the SLOCC invariant. In Ref. [21] a state having four different roots was shown be SLOCC equivalent to a state within the one-parameter family: $|\Psi(\mu)\rangle = |\mathrm{GHZ}_4\rangle + \mu|D_4^{(2)}\rangle$ with $\mu \in \mathbb{C}\backslash\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\}$ [where $|\mathrm{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(|s=2, m=-2\rangle + |s=2, m=2\rangle)$ and $|D_4^{(2)}\rangle = |s=2, m=0\rangle$]. Computing the cross ratio for this family, one obtains the relation $\lambda = \frac{1}{2}(\sqrt{3}\mu + 1)$.

For $n > 4$, the set of permutation related cross ratios leads to complex geometrical patterns and the identification of a canonical domain analogous to $\mathcal{D}_1$ is difficult (as an example for $n = 5$ all 5! transformations lead to inequivalent cross-ratio sets). We therefore introduce a more symmetrical formulation of the invariant quantities, $I_k(\boldsymbol{\lambda}) = \sum_\Pi [\lambda'_1(\Pi)]^k$, which amounts to sum the kth powers of the transformed cross ratios (say of λ'_1) over the complete orbit of the permutation group. Back to $n = 4$, a nontrivial symmetrized invariant $I_2(\lambda)$ is obtained: $I_2(\lambda) = \frac{2(\lambda^6+1)-6(\lambda^5+\lambda)+9(\lambda^4+\lambda^2)-8\lambda^3}{(\lambda-1)^2\lambda^2} = -3 + \frac{27}{2}J(\lambda)$, where $J(\lambda)$ is known as the Klein modular invariant [22]. The next case is $n = 5$, where two independent invariants $I_2(\lambda_1, \lambda_2)$ and $I_4(\lambda_1, \lambda_2)$ can be generated by summing the cross ratios squares and fourth powers over the 120 permutations. Because of a lack of space, the explicit form of the two invariants is not given here. When two Majorana roots are equal one can, without loss of generality, let λ_1 go to zero, in which case both invariants diverge, but we find again the Klein invariant in the following expression $\lim_{\lambda_1\rightarrow 0} \frac{I_4(\lambda_1,\lambda_2)}{I_2(\lambda_1,\lambda_2)^2} = \frac{1}{8} - \frac{2}{27J(\lambda_2)}$, which allows us to fully characterize the states having 3 or 4 different roots.

In conclusion, we have explicitly constructed a set of entanglement invariants under LOCC and SLOCC for symmetric n-qubit states and given several examples for n up to five. We also expect that this correspondence between ILO and Möbius transformations, may find further possible experimental consequences. Indeed, a generic Möbius transform can be decomposed into elementary operations, such as translations, rotations, inversions, and dilation. It would therefore be very interesting to perform such elementary operations by implementing suitable positive operator valued measures within the symmetric sector.

P. R. acknowledges support through FCT BPD Grant No. SFRH/BPD/43400/2008. R. M. also acknowledges discussions with M. Kus and K. Zyczkowski about the unit 3-tangle manifold and with J-M Maillard, about SLOCC invariants, who, in particular, noted the connection between $I(\lambda)$ and the Klein invariant.

*ribeiro@cfif.ist.utl.pt
†remy.mosseri@upmc.fr

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge Univ. Press, Cambridge, 2000).
[2] O. Gühne and G. Tóth, Phys. Rep. **474**, 1 (2009).
[3] M. Plenio and S. Virmani, Quantum Inf. Comput. **7**, 1 (2007).
[4] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. A **63**, 012307 (2000).
[5] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[6] M. Grassl, M. Rötteler, and T. Beth, Phys. Rev. A **58**, 1833 (1998).
[7] J.-G. Luque and J.-Y. Thibon, Phys. Rev. A **67**, 042303 (2003).
[8] P. Levay, J. Phys. A **38**, 9075 (2005).
[9] R. Hübener, M. Kleinmann, T.-C. Wei, C. González-Guillén, and O. Gühne, Phys. Rev. A **80**, 032324 (2009).
[10] D. Li, X. Li, H. Huang, and X. Li, Europhys. Lett. **87**, 20006 (2009).
[11] L. Chen, A. Xu, and H. Zhu, Phys. Rev. A **82**, 032301 (2010).
[12] J. Martin, O. Giraud, P. A. Braun, D. Braun, and T. Bastin, Phys. Rev. A **81**, 062347 (2010).
[13] D. J. H. Markham, arXiv:1001.0343.
[14] C. Cenci, D. Lyons, L. Snyder, and S. Walck, Quantum Inf. Comput. **10**, 1029 (2010).
[15] M. Aulbach, D. Markham, and M. Murao, New J. Phys. **12**, 073025 (2010).
[16] M. Aulbach, D. Markham, and M. Murao, *Geometric Entanglement of Symmetric States and the Majorana Representation*, Lecture Notes in Computer Science Vol. 6519 (Springer, New York, 2011), pp. 141–158.
[17] W. Wieczorek, C. Schmid, N. Kiesel, R. Pohlner, O. Gühne, and H. Weinfurter, Phys. Rev. Lett. **101**, 010503 (2008).
[18] T. Bastin, C. Thiel, J. von Zanthier, L. Lamata, E. Solano, and G. S. Agarwal, Phys. Rev. Lett. **102**, 053601 (2009).
[19] G. Tóth and O. Gühne, Phys. Rev. Lett. **102**, 170503 (2009).
[20] E. Majorana, Nuovo Cimento **9**, 43 (1932).
[21] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, Phys. Rev. Lett. **103**, 070503 (2009).
[22] F. Klein, *Gesammelte mathematische Abhandlungen* (Springer, Berlin, 1921–1923), Vol. 3.
[23] P. Ribeiro, J. Vidal, and R. Mosseri, Phys. Rev. Lett. **99**, 050402 (2007).
[24] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[25] I. Bengtsson and K. Zyczkowski, *Geometry of Quantum States* (Cambridge University Press, Cambridge, U.K., 2006).
[26] J. Kempe, Phys. Rev. A **60**, 910 (1999).
[27] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
[28] G. Sarbicki, Open Syst. Inf. Dyn. **12**, 143 (2005).
[29] P. Mathonet, S. Krins, M. Godefroid, L. Lamata, E. Solano, and T. Bastin, Phys. Rev. A **81**, 052315 (2010).
[30] A. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin, Heidelberg, New York, 1986).