Negative Effective Gravity in Water Waves by Periodic Resonator Arrays

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Based on analytic derivations and numerical simulations, we show that near a low resonant frequency water waves cannot propagate through a periodic array of resonators (bottom-mounted split tubes) as if water has a negative effective gravitational acceleration g_e and positive effective depth h_e. This gives rise to a low-frequency resonant band gap in which water waves can be strongly reflected by the resonator array. For a damping resonator array, the resonant gap can also dramatically modify the absorption efficiency of water waves. The results provide a mechanism to block water waves and should find applications in ocean wave energy extraction.

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Water waves are mechanical waves that propagate along the interface between water and air, and the restoring force is provided by gravity [1]. For a constant water depth h, the dispersion of linear water waves is given by

$$\omega^2 = gk\tanh(kh), \tag{1}$$

where ω is the angular frequency, $k \equiv 2\pi/\lambda$ is the wave number, λ is the wavelength, and g is the gravitational acceleration [2]. This dispersion relationship, first derived by Airy in 1841 [3], is valid for various water depths and forms the basis of modern hydrodynamics and ocean engineering [1,2].

Recently, the interaction of water waves and periodic structures, such as rippled bottoms and periodic vertical obstacle arrays, has attracted considerable attention [4–18]. It is found that while the dispersions of water waves can be strongly modified by periodic structures, they remain simple in the long-wavelength range [6–14]. In particular, when the wavelength is longer than fourfold of the periodic length a ($\lambda > 4a$), the periodic system can be viewed as a homogeneous liquid with an effective gravitational acceleration g_e and effective depth h_e and thus possesses a simple dispersion

$$\omega^2 = g_e k_e \tanh(k_e h_e), \tag{2}$$

where k_e is the effective wave number [15]. The effective parameters (g_e and h_e) can be different from the values (g and h) of the water without structures (background), resulting in a new type of water-wave refraction [16,17]. However, the effective parameters have the same positive signs as those of the background. As a result, long water waves can propagate through periodic structures [4–18], and this is a reason why tsunamis are difficult to block by periodic structures.

In this Letter, we show both analytically and numerically that, near a resonant frequency, long water waves cannot propagate through a periodic array of resonators as if the system has a negative effective gravitational acceleration ($g_e < 0$) and positive effective depth. This gives rise to a low-frequency resonant band gap in which water waves are strongly reflected by the resonator array. Although the results are demonstrated by using resonators of vertical bottom-mounted split tubes, they can be realized by other resonators such as damping buoys for ocean wave energy conversion [19–23]. For damping resonator arrays, it is found that the resonant band gap can dramatically modify the absorption efficiency of water waves.

We consider linear, inviscid, and irrotational water waves in an infinite extent of water of constant depth h, pierced with bottom-mounted, identical, vertical, split rigid tubes as shown in Fig. 1(a). The tubes have an outer radius r_1, inner radius r_2, and total split width Δ and are arranged in a square lattice with lattice constant a. Set $\mathbf{r} = (x, y)$ in the horizontal plane and z as the vertical axis. For harmonic water waves, the vertical displacement of the water surface η is related to a potential φ through $\eta(\mathbf{r}, t) = \mathrm{Re}[-\frac{i\omega}{g}\varphi(\mathbf{r})e^{-i\omega t}]$. φ satisfies the two-dimensional Helmholtz equation [1]

$$\nabla^2\varphi + k^2\varphi = 0, \tag{3}$$

subjected to a no-flow condition at the surface of each tube, namely, $\mathbf{n}\cdot\nabla\varphi = 0$ with $\mathbf{n}$ normal to the tube surface.

We will first derive analytic formulas for the effective parameters of the periodic system with the coherent-potential-approximation method [15,24]. We consider a circular water column with radius $R = a/\sqrt{\pi}$ [so that $\pi r_1^2/(\pi R^2)$ equals the filling fraction $f_s \equiv \pi r_1^2/a^2$] and water depth h and pierced by a split rigid tube with radius r, surrounded by the effective liquid with parameters

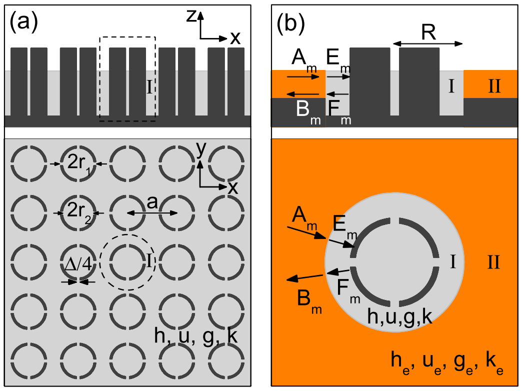

FIG. 1 (color online). Schematic diagrams of a periodic array of identical, vertical, split rigid tubes standing in water with constant depth h (a) and a water column pierced by a vertical split rigid tube, which is surrounded by an effective liquid (b). The top panels are side views, and the bottom panels are vertical views.

(g_e and h_e) as shown in Fig. 1(b). By using the cylindrical coordination (ρ, ϕ) with the origin at the center of the tube, φ can be written as

$$\varphi^{\mathrm{I}} = \sum_m [E_m J_m(k\rho) + F_m H_m(k\rho)]e^{im\phi} \quad \text{for } R \geq \rho \geq r_1,$$

$$\varphi^{\mathrm{II}} = \sum_m [A_m J_m(k_e\rho) + B_m H_m(k_e\rho)]e^{im\phi} \quad \text{for } \rho \geq R.$$

Here, the Bessel function J_m and Hankel function H_m stand for the incident and scattering waves, respectively, and k_e is the wave number in the effective liquid. At the boundary of the water column, the potential φ and flow $u\nabla\varphi$ should be continuous [$\varphi^{\mathrm{I}}(R) = \varphi^{\mathrm{II}}(R)$ and $u\frac{\partial\varphi^{\mathrm{I}}(R)}{\partial\rho} = u_e\frac{\partial\varphi^{\mathrm{II}}(R)}{\partial\rho}$] with the reduced depths being [25]

$$u = [\tanh(kh)]/k, \qquad u_e = [\tanh(k_e h_e)]/k_e. \tag{4}$$

It can thus be shown that the water column will not scatter the zeroth-order and first-order cylindrical waves ($B_0, B_1 = 0$, which defines the effective liquid) when

$$-\frac{ukJ_0'(kR) - J_0(kR)T_0(u_e, k_e, R)}{ukH_0'(kR) - H_0(kR)T_0(u_e, k_e, R)} = D_0, \tag{5}$$

$$-\frac{ukJ_1'(kR) - J_1(kR)T_1(u_e, k_e, R)}{ukH_1'(kR) - H_1(kR)T_1(u_e, k_e, R)} = D_1, \tag{6}$$

where $T_{0,1}(u_e, k_e, R) = u_e k_e J_{0,1}'(k_e R)/J_{0,1}(k_e R)$ and $D_m \equiv F_m/E_m$ is the scattering coefficients of the split tube. We note that for a single split tube with the incidence of a plane wave, the total scattering and absorption cross sections are [26], respectively,

$$C_{\mathrm{sct}} = \frac{2\lambda}{\pi}\sum_m |D_m|^2 \equiv C_{\mathrm{sct},m},$$

$$C_{\mathrm{abs}} = \frac{\lambda}{2\pi}\sum_m (1 - |2D_m + 1|^2).$$

Using the restriction $|2D_m + 1| \leq 1$, one can infer that the maximal contributions of a single channel to the scattering and absorption cross sections are $2\lambda/\pi$ and $\lambda/(2\pi)$, respectively [27]. Once the scattering coefficients (D_0 and D_1) are known, the effective parameters (u_e and k_e) can be solved numerically from Eqs. (5) and (6), and g_e and h_e can then be obtained by Eqs. (2) and (4).

For long water waves ($kR, k_e R \ll 1$) [25], the effective parameters (u_e and k_e) can be obtained by the reduced forms of Eqs. (5) and (6):

$$\frac{k_e}{k} = \sqrt{1 + pD_0}\sqrt{\frac{1 + pD_1}{1 - pD_1}}, \qquad \frac{u_e}{u} = \frac{1 - pD_1}{1 + pD_1}, \tag{7}$$

where $p = 4f_s/(i\pi k^2 r_1^2)$ [25]. For an array of bottom-mounted rigid cylinders ($-D_0 \approx D_1 \approx \frac{i}{4}\pi k^2 r_1^2$) [10], we have $k_e/k = \sqrt{1 + f_s}$and $u_e/u = \frac{1-f_s}{1+f_s}$, consistent with our previous derivations [15]. We note that the index $n \equiv k_e/k$ is important for describing the refraction of water waves by periodic structures [15].

To calculate the scattering coefficients of a bottom-mounted rigid split tube, we replace the tube wall by its effective liquid with a thickness of $(r_1 - r_2)/n_t$, a wave number of $n_t k$, unchanged g, and $n_t = 2\pi r_1/\Delta$ [28,29]. This replacement is valid for wavelengths much longer than the slit width ($k\Delta/4 \ll 1$). For longer wavelengths ($kR, k_e R \ll 1$) [25], the scattering coefficients can be expressed as [29]

$$D_0 \approx \frac{i}{4}\pi k^4 r_1^2/[k_R^2 - k(k + i\Gamma)], \quad D_1 \approx \frac{i}{4}\pi k^2 r_1^2. \tag{8}$$

Here Γ represents the loss of the resonance, and the resonant wave number is given by

$$k_R \equiv 2\pi/\lambda_R = \sqrt{\Delta/[\pi r_2^2(r_1 - r_2)]}. \tag{9}$$

We note that Eq. (9) can also be obtained by a spring (water in the tube) and mass (water in the slits) model. Consequently, analytic formulas can be obtained for the effective parameters of the split-tube array system:

$$\frac{k_e}{k} = \sqrt{\left(1 + \frac{f_s k^2}{k_R^2 - k(k + i\Gamma)}\right)\frac{1 + f_s}{1 - f_s}}, \quad \frac{u_e}{u} = \frac{1 - f_s}{1 + f_s}. \tag{10}$$

In a wave number range above the resonance ($k_R < k < k_+$), k_e becomes a complex number giving rise to a band gap of water wave. Here the upper gap edge is given by

$$k_+ \equiv 2\pi/\lambda_+ = k_R/\sqrt{1 - f_s}. \tag{11}$$

In Fig. 2, we demonstrate the resonant gap using split tubes with parameters ($r_1 = 0.36a$, $r_2 = 0.32a$, and

$\Delta = 0.007a$). For a single split tube, the scattering cross section is found to be maximized at the frequency ($a/\lambda_R = 0.11$) due to an $m = 0$ resonance [Fig. 2(a)]. Consequently, the effective wave number of the split-tube array has a nonzero imaginary part in a frequency range above the resonance ($0.111 < a/\lambda < 0.143$) [Fig. 2(b)]. It is interesting to note that, in the resonant gap, a negative effective acceleration of gravity ($g_e < 0$) occurs [Fig. 2(c)] while the effective reduced depth remains positive ($u_e = 0.24u$).

To verify the resonant gap, we do multiple-scattering simulations for impinging of plane water waves upon a five-layer split-tube array. We note that the multiple-scattering method includes high-order cylindrical waves [10,18] and thus can reproduce experimental results well [14,15]. In the band gap, water waves are found to be completely reflected by the structure [Fig. 2(d)]. The lower gap edge persists at the resonant frequency, and the upper gap edge increases with increasing the filling fraction of tubes, agreeing well with the analytic results [Fig. 3]. We note that, since Eq. (3) also occurs in acoustics, a similar resonant gap (with a negative effective modulus) was observed experimentally in acoustic resonator arrays [30].

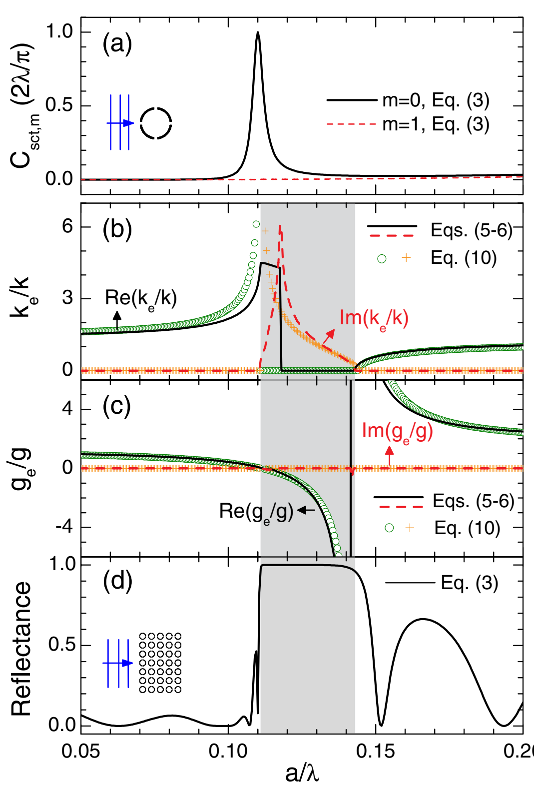

FIG. 2 (color online). (a) Partial water-wave scattering cross section of a single, bottom-mounted, split rigid tube. (b) Effective wave number k_e and (c) effective gravitational acceleration g_e for an infinite periodic split-tube array system. (d) Reflectance for normal incidence of a plane water wave with wavelength λ upon a five-layer split-tube array. The parameters of the tubes are $r_1 = 0.36a$, $r_2 = 0.32a$, and $\Delta = 0.007a$.

By now we have shown that, by using an array of resonators, a resonant gap far below conventional Bragg gaps (with central wavelengths $\lambda = 2a/L$ and L being an integer) can be achieved for water waves. To obtain such a low-frequency gap, three criteria should be met. (i) The resonators have either a monopole ($m = 0$) resonance or a dipole ($m = 1$) resonance. When a monopole (dipole) resonance exists, a negative g_e (h_e) can be achieved in the gap. (ii) The resonant wavelength is much larger than the horizontal size of a single resonator. (iii) The filling fraction of resonators is efficiently large (> 0.2). We note that since conditions (ii) and (iii) were not met, the resonant gap has not been discovered in previous studies on the interaction of water waves and resonators [22,23].

Damping resonators, such as heaving buoys, can be applied to extract the ocean wave energy [19–21], and an array of damping resonators is regarded as a key part of future ocean wave power plants [22,23]. Here, we investigate the influence of the above resonant gap on the absorption spectrum of damping resonator arrays. As a demonstration, we consider damping split tubes with $\mathrm{Im}(k) = 2 \times 10^{-3}\pi/a$ in the split regions. Although a single split tube has a moderate scattering cross section [$C_{\mathrm{sct}} = 0.47(2\lambda/\pi)$] at the resonant frequency [Fig. 4(a), dashed line], high reflection ($> 90\%$) is still observed in the gap for a five-layer split-tube array [Fig. 4(b), dashed line]. Because of the $m = 0$ resonance, the total absorption cross section of a single split tube exhibits a peak at the resonant frequency [Fig. 4(a), solid line]. However, two absorption peaks are observed for a five-layer split-tube array [Fig. 4(b), solid line], indicating a strong modification of the density of resonant modes by the gap. Because of the peak at the upper gap edge, the absorption integrated over the whole spectrum is not reduced significantly by the gap.

In summary, we have demonstrated that the long water waves propagate through an array of bottom-mounted split tubes as if water has an effective gravitational acceleration g_e and effective depth h_e given by Eqs. (2), (4), and (10). A low-frequency resonant gap is found where a negative g_e

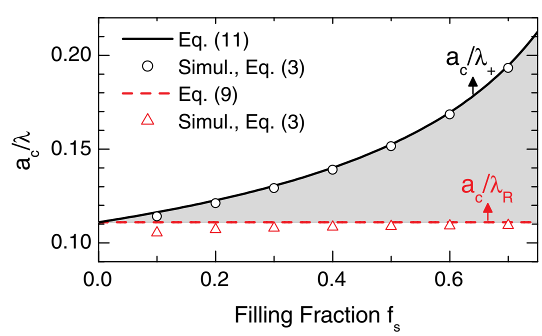

FIG. 3 (color online). The water-wave gap edges of periodic split-tube arrays as functions of the filling fraction $f_s \equiv \pi r_1^2/a^2$. The parameters of the tubes are $r_1 = 0.36a_c$, $r_2 = 0.32a_c$, and $\Delta = 0.007a_c$.

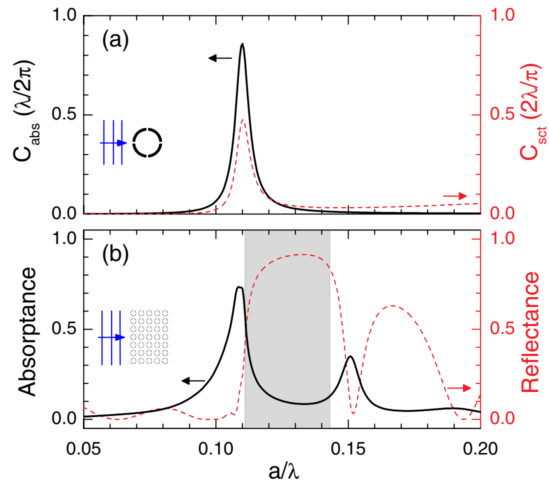

FIG. 4 (color online). (a) Total water-wave absorption and scattering cross sections of a single, bottom-mounted, split rigid tube. (b) Absorptance and reflectance for normal impinging of a plane water wave with wavelength λ upon a five-layer split-tube array. The tubes have the same parameters as those in Fig. 2. The absorption is introduced by using $\mathrm{Im}(k) = 2 \times 10^{-3}\pi/a$ in the split regions. The lines are simulation results based on Eq. (3).

and positive h_e occur and the propagation of water waves is forbidden. For a damping resonator array, the absorption of water waves is found to be greatly enhanced at the edges of the resonant gap. Our results provide a new mechanism for the formation of water-wave band gaps and should be useful to engineers on ocean wave energy extraction.

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