Asymmetric Wave Propagation in Nonlinear Systems

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A mechanism for asymmetric (nonreciprocal) wave transmission is presented. As a reference system, we consider a layered nonlinear, nonmirror-symmetric model described by the one-dimensional discrete nonlinear Schrödinger equation with spatially varying coefficients embedded in an otherwise linear lattice. We construct a class of exact extended solutions such that waves with the same frequency and incident amplitude impinging from left and right directions have very different transmission coefficients. This effect arises already for the simplest case of two nonlinear layers and is associated with the shift of nonlinear resonances. Increasing the number of layers considerably increases the complexity of the family of solutions. Finally, numerical simulations of asymmetric wave packet transmission are presented which beautifully display the rectifying effect.

The design of small devices to control energy and/or mass flows at different spatial scales is a suggestive challenge from both a theoretical and applied viewpoint. Since the first proposal of a thermal diode [1], capable of transmitting heat asymmetrically between two temperature sources, several studies appeared in the literature [2], including some experimental realizations [3,4].

A related issue concerns the possibility of devising a ''wave diode'' in which electromagnetic or elastic waves are transmitted differently along two opposite propagation directions. A so-called optical diode has been proposed in Ref. [5,6] and later on discussed both theoretically [7,8] and experimentally [9]. There is also a recent proposal for a diode based on left-handed metamaterials [10]. Another domain of application is the propagation of acoustic pulses in granular systems. Indeed, experimental studies demonstrated a change of solitary wave reflectivity from the interface of two granular media [11].

The basic idea for a ''wave diode'' was proposed for nonlinear photonic crystals [7]: the rectification depends on whether the second harmonic of the fundamental wave is transmitted or not. As a definition of a rectifying device we would instead propose that the transmitted power at fixed incident amplitude and at the same frequency ω should be sensibly different in the two opposite propagation directions. In the present Letter we pursue a novel approach that exploits distinctive features of nonlinear dynamical systems such as multistabiliy and amplitude-dependent resonances to achieve such an effect.

In a linear, time-reversal symmetric system this possibility is forbidden by the reciprocity theorem [12]. Therefore, one needs to consider nonlinear and asymmetric systems [13]. As a reference model we will focus on the discrete nonlinear Schrödinger (DNLS) equation [14,15]

with spatially varying coefficients. It has been demonstrated [16] that DNLS can be a sensible approximation for the evolution of longitudinal Bloch waves in layered photonic or phononic crystals (Fig. 1). Variable coefficients describe different nonlinear properties of each layer and the presence of defects. Beyond its relevance in many different physical contexts, the DNLS equation has the big advantage of being among the simplest dynamical systems amenable to a complete theoretical analysis. For our purpose, it is particularly convenient as it allows us to solve the scattering problem exactly without the complications of having to deal with wave harmonics.

More precisely, let us consider the stationary DNLS equation defined on an infinite one-dimensional lattice

$$\omega \psi_n = V_n \psi_n - \psi_{n+1} - \psi_{n-1} + \alpha_n |\psi_n|^2 \psi_n. \qquad (1)$$

We will assume the usual scattering setup where V_n and α_n are nonvanishing only for $1 \leq n \leq N$. The two
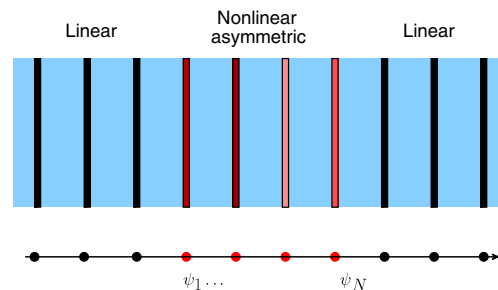

FIG. 1 (color online). Sketch of a layered photonic or phononic system. The central N layers are nonlinear, nonmirror symmetric with respect to the structure center. In the limit of a vanishing width of the thinner layers the dynamics within high-frequency Bloch bands is described by a DNLS equation for the envelope ψ_n (see Refs. [16,18] for details).

semi-infinite portions ($n < 1$, $n > N$) of the lattice, model two leads where the wave can propagate freely.

Let us look for solutions of the associated transmission problem

$$\psi_n = \begin{cases} R_0 e^{ikn} + R e^{-ikn} & n \leq 1 \\ T e^{ikn} & n \geq N \end{cases} \quad (2)$$

where $\omega = -2\cos k$ and $0 \leq k \leq \pi$ for the wave coming from the left direction; R_0, R and T are the incident, reflected and transmitted amplitudes, respectively. The solution sought must be complex in order to carry a non-vanishing current $J = 2|T|^2 \sin k$.

To break the mirror symmetry with respect to the center of the nonlinear portion, one must choose at least one of the two sets of coefficients V_n, α_n such that $V_n \neq V_{N-n+1}$, $\alpha_n \neq \alpha_{N-n+1}$. Note that the transmission of the right-incoming wave with the same R_0 and ω is computed by solving the problem with $(V_n, \alpha_n) \to (V_{N-n+1}, \alpha_{N-n+1})$ (i.e., "flipping the sample"). In the following, we will refer to such solutions as those having negative wave numbers, $-k$. Nonlinearity is essential as for $\alpha_n = 0$ the transmission coefficient is the same for waves coming from the left or right side, independently on V_n. This is due to time-reversal invariance of the underlying equations of motion [17].

The standard way to solve the problem is to introduce the (backward) transfer map [18–21]

$$u_{n-1} = -v_n + (V_n - \omega + \alpha_n |u_n|^2) u_n, \qquad v_{n-1} = u_n \quad (3)$$

where $u_n = \psi_n$ and $v_n = \psi_{n+1}$. Note that these are complex quantities; therefore, the map is nominally four dimensional. However, due to conservation of energy and norm, it can be reduced to a two-dimensional area-preserving map [18–21] with an additional control parameter (the conserved current J). The solutions are found by iterating (3) from the initial point $u_N = T \exp(ikN)$, $v_N = T \exp[ik(N+1)]$ dictated by the boundary conditions of Eq. (2). For fixed T and k, the incident and reflected amplitudes are determined as $R_0 = \frac{\exp(-ik)u_0 - v_0}{\exp(-ik) - \exp(ik)}$, $R = \frac{\exp(ik)u_0 - v_0}{\exp(ik) - \exp(-ik)}$ and the transmission coefficient is $t(k, |T|^2) = |T|^2/|R_0|^2$. Note that if $(u_0, v_0) = (u_N, v_N)$ (periodic point of the map) then $t = 1$.

In order to illustrate the effect in the simplest case we consider the DNLS dimer $N = 2$. The coefficient t can be thereby computed analytically iterating the map twice:

$$t = \left| \frac{e^{ik} - e^{-ik}}{1 + (\nu - e^{ik})(e^{ik} - \delta)} \right|^2 \quad (4)$$

where $\delta = V_2 - \omega + \alpha_2 T^2$, $\nu = V_1 - \omega + \alpha_1 T^2 [1 - 2\delta \cos k + \delta^2]$. The formula applies for $k > 0$. As explained above, to solve for the case $k < 0$ one has to exchange the subscripts 1 and 2 which is equivalent to reverse the sample.

Up to now no hypothesis has been made on the coupling coefficients. For simplicity, we impose $\alpha_{1,2} = \alpha > 0$ henceforth and let $V_{1,2} = V_0(1 \pm \varepsilon)$. In the symmetric case $\varepsilon = 0$, it can be verified that transmission is unity for $V_0 + \alpha T^2 = 0$ (for $V_0 < 0$) and $V_0 + \alpha T^2 = \omega$ (for $V_0 < \omega$). These nonlinear resonances can be regarded as the continuation of the extended (i.e., perfectly transmitting) states of the corresponding linear problem [22] to nonvanishing α. As a result, the transmission curves ($|R_0|^2$, $|T|^2$) are tangent to the bisectrix in two points where $t = 1$ (dashed lines in Figs. 2(a) and 2(b), respectively).

For the asymmetric case, $\varepsilon \neq 0$, the resonances are detuned differently for the $k > 0$ and $k < 0$ cases, leading to a nonreciprocal transmission [see the solid lines in Figs. 2(a) and 2(b)] with maximal t smaller than 1. In particular, there exist two windows $W_{1,2}$ of amplitude of the incident wave R_0 where we have three solutions for $k > 0$ and only one for $k < 0$. Here the asymmetry becomes maximal. Figure 3 illustrates the different types of solutions in the region W_1 [see the points in the inset of Fig. 2(a)]. The two low-amplitude solutions are very similar, as expected for weak nonlinearity [23].

Since the phenomenon is of nonlinear origin the asymmetry depends on both frequency and amplitude. To quantify its efficiency, in Fig. 4 we report the rectifying factor

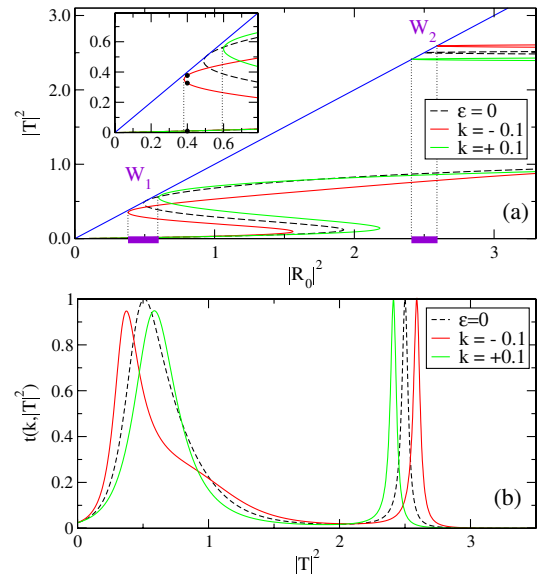$$f = \frac{t(k, |T|^2) - t(-k, |T|^2)}{t(k, |T|^2) + t(-k, |T|^2)} \quad , \quad (5)$$

FIG. 2 (color online). DNLS dimer, $N = 2$, $V_0 = -2.5$, $\alpha = 1$, $|k| = 0.1$. Comparison between the symmetric case ($\varepsilon = 0$, dashed lines) and the asymmetric one ($\varepsilon = 0.05$, solid lines). (a) Transmission curves; the inset is an enlargement of the low-amplitude region. The vertical lines mark the turning points of the curves. Accordingly, the heavy lines on the horizontal axis are the multistability windows $W_{1,2}$ where the diode effect maximally occurs. (b) Transmission coefficients as a function of the transmitted intensity.
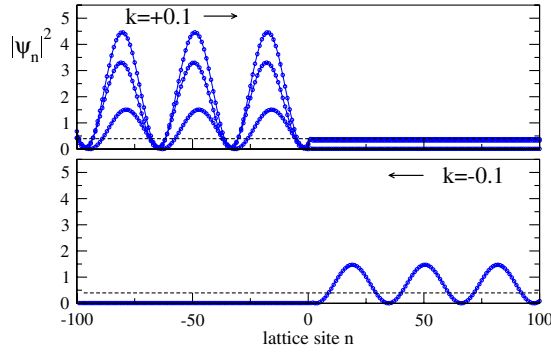
FIG. 3 (color online). Square modulus of the solutions corresponding to the same incident amplitude $|R_0|^2 = 0.4$ (marked by full dots in the inset of Fig. 2), other parameters as in previous figure. Upper panel: The three left-propagating solutions corresponding to $|T|^2 = 0.327, 0.377, 0.01$ (top to bottom). Lower panel: The right-propagating solution corresponding to $|T|^2 = 0.01$. The oscillations are caused by the interference between the incident and reflected waves (wave number $2k$).

which approaches ± 1 for maximal asymmetry. Note that, although increasing ε broadens the regions in which $|f|$ is relatively large, the overall transmitted intensity is reduced as well.

Increasing the number N of nonlinear layers will considerably increase the complexity of the transmission patterns. Indeed, for large N, the transmitting (resp. nontransmitting) solutions correspond to bounded (resp. escaping) orbits of the map (3). Because of the mixed phase space, t is nonvanishing only on a fractal set of the parameter space [19]. The system thus exhibits a complicated form of multistability, meaning that for a fixed input one may have (infinitely) many outputs [18,19]. We thus expect that the structure of the windows W_n will approach that of a Cantor set in the large N limit. Indeed, Fig. 5 indicates that even for small N, the W_n shrink and
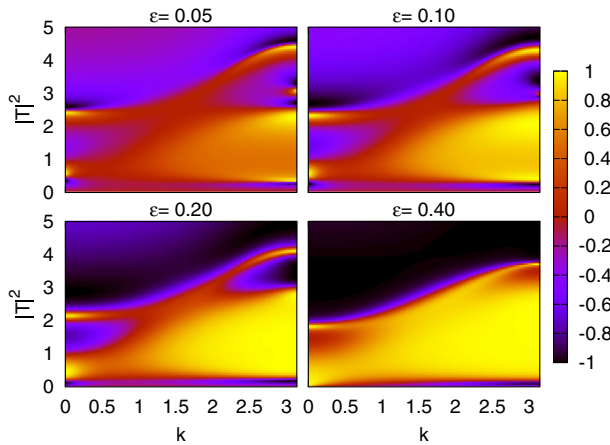

FIG. 4 (color online). Contour plot of the rectifying factor (5) for increasing asymmetry level ε. Other parameters as in previous figure.

the texture of the regions where $|f| \lesssim 1$ complicates considerably.

What are the consequences of the above results on the transmission of wave packets? In a nonlinear system where the superposition principle no longer holds, the connection between the two problems is not trivial. To address this problem, we solved numerically the time-dependent DNLS

$$i\dot{\phi}_n = V_n\phi_n - \phi_{n+1} - \phi_{n-1} + \alpha_n|\phi_n|^2\phi_n \qquad (6)$$

on a finite lattice $|n| \leq M$ with open boundary conditions, for the case of the dimer discussed above. We take as the initial condition a Gaussian wave packet

$$\phi_n(0) = I \exp\left[-\frac{(n - n_0)^2}{w} + ik_0 n\right]. \qquad (7)$$

The upper panels of Fig. 6 display the evolution of two packets with the same I and opposite wave number k_0 impinging on the nonlinear dimer. The asymmetry of their propagation is manifest. Here, the parameters have been chosen empirically to obtain the maximal asymmetry. More precisely, we first measured the wave packet transmission coefficient t_p, defined as the ratio between the transmitted norm $\sum_{n>N}|\phi_n|^2$ at the end of the run divided
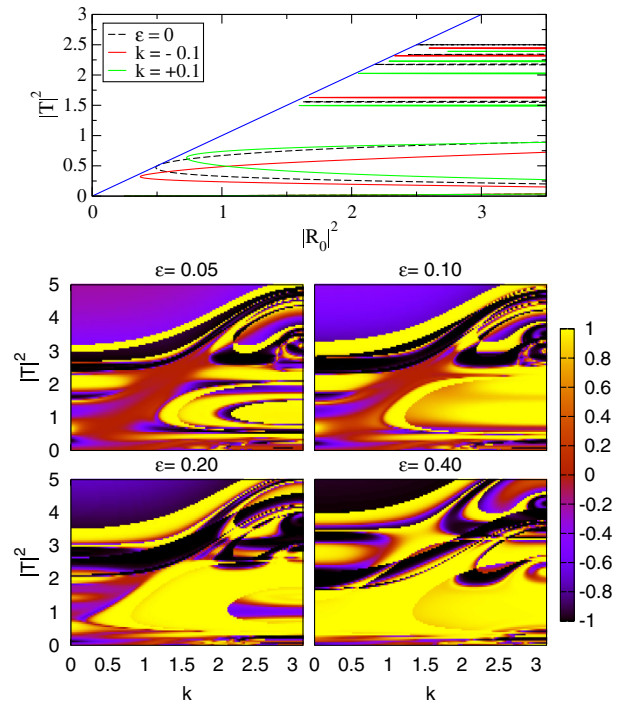

FIG. 5 (color online). Upper panel: Transmission curves for DNLS with $N = 4$, linearly modulated parameter $V_n = V_0[1 + \varepsilon\{1 - 2(n - 1)\}/(N - 1)]$, other parameters as in the previous figures. Comparison between the symmetric case ($\varepsilon = 0$, dashed lines) and the asymmetric one ($\varepsilon = 0.05$, solid lines). Lower panels: Contour plots of the rectifying factor (5) for increasing asymmetry. Similar results are obtained for different types of modulation.
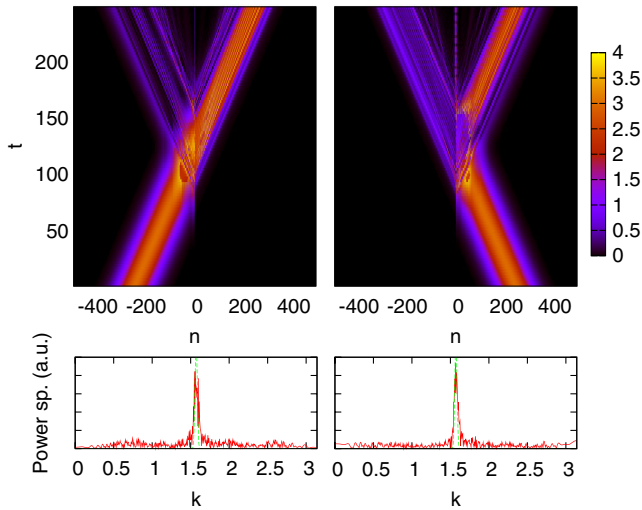
FIG. 6 (color online).   Numerical simulations of the propagation of Gaussian wave packets, Eq. (7) impinging on a DNLS dimer. Here $V_0 = -2.5$, $|k_0| = 1.57$, $\varepsilon = 0.05$, $M = 500$, $|I|^2 = 3$, $w = 10^4$ and $n_0 = \mp 250$, respectively. Lower panels: Power spectra of the real part of ϕ_n at times $t = 0$ (green dashed line) and $t = 250$ (red solid line).

by the initial one $\sum_{n<0} |\phi_n(0)|^2$, as a function of $|I|^2$. The data of Fig. 6 correspond to the amplitude I for which the transmission is maximal for the left-incoming packet ($t_p \simeq 0.8$) and minimal for the right-incoming one ($t_p \simeq 0.3$). Although the packets are significantly distorted after scattering, the Fourier analysis shows that they remain almost monochromatic at the incident wave number k_0 (lower panels of Fig. 6). It is also noteworthy that reflection is associated with the creation of a localized excitation, strongly reminiscent of the nonlinear impurity modes [15].

In conclusion, based on the DNLS model, we have demonstrated a mechanism which leads to nonreciprocal wave transmission. The new class of solutions found here are of interest both theoretically as well as to envisage possible experimental realizations in nonlinear layered photonic or phononic systems. To the extent to which the DNLS can be considered a realistic model for such media, our results may open the way to novel strategies to control and optimize wave propagation and to design devices for sound or light rectification.

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