# Surprises in the Evaporation of 2D Black Holes

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Quantum evaporation of Callan-Giddings-Harvey-Strominger black holes is analyzed in the mean-field approximation, incorporating backreaction. Detailed analytical and numerical calculations show that, while some of the assumptions underlying the standard evaporation paradigm are borne out, several are not. Furthermore, if the black hole is initially macroscopic, the evaporation process exhibits remarkable universal properties (which are distinct from the features observed in the simplified, exactly soluble models). Finally, our results provide support for the full quantum gravity scenario recently developed by Ashtekar, Taveras, and Varadarajan.

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*Introduction.*—Since the early 1990s, a number of two-dimensional (2D) black hole models have been studied to gain further insight into the quantum dynamics of black hole evaporation. Physically, the most interesting among them is due to Callan-Giddings-Harvey-Strominger (CGHS) [1]. Simplified versions of this model are exactly soluble but also have important limitations discussed, e.g., in Refs. [2,3]. Therefore, results obtained in those models are not reliable indicators of what happens in the full CGHS dynamics. In this Letter, we present key results from a new analysis of CGHS black holes using a mean-field or semiclassical approximation. These findings are surprising in two respects. First, several features of the standard CGHS paradigm [2] of quantum evaporation are not realized. Second, black holes resulting from a prompt collapse of a large Arnowitt-Deser-Misner (ADM) mass exhibit rather remarkable behavior: After an initial transient phase, dynamics of various physically interesting quantities at right future null infinity $\mathcal{I}_R^+$ flow to *universal curves*, independent of the details of the initial collapsing matter distribution. This universality strongly suggests that information in the collapsing matter on $\mathcal{I}_R^-$ *cannot* in general be recovered at $\mathcal{I}_R^+$. However, we also find strong evidence supporting the scenario of Ref. [4] in which the $S$ matrix from (left past infinity) $\mathcal{I}_L^-$ to $\mathcal{I}_R^+$ is unitary. This distinction between unitarity and information recovery is a peculiarity of 2D.

In this Letter, we summarize the main results. An extensive treatment can be found in Ref. [5], details of the numerics in Ref. [6], and a thorough investigation of the full quantum issues in Ref. [7].

*Model.*—In the CGHS model, geometry is encoded in a physical metric $g$ and a dilaton field $\phi$, coupled to $N$ massless scalar fields $f_i$. Since we are in 2D with $\mathbb{R}^2$ topology, we can fix a fiducial flat metric $\eta$ and write $g$ as $g^{ab} = \Omega \eta^{ab}$. Then it is convenient to describe geometry through $\Phi := e^{-2\phi}$ and $\Theta := \Omega^{-1}\Phi$. The model has 2 constants: $\kappa$ and $G$ with dimensions $[L]^{-1}$ and $[ML]^{-1}$.

Our investigation is carried out within the mean-field approximation (MFA) of Refs. [4,7] in which one ignores quantum fluctuations of geometry but not of matter. To ensure a sufficiently large domain of validity, we must have large $N$ and we assume that each scalar field $f_i$ has the same profile. Black hole formation and evaporation is described entirely in terms of nonlinear partial differential equations. Denote by $z^\pm$ the advanced and retarded null coordinates of $\eta$ so that $\eta_{ab} = 2\partial_{(a} z^+ \partial_{b)} z^-$. We will set $\partial_\pm \equiv \partial/\partial z^\pm$. Then we have the evolution equations

$$\Box_{(\eta)} f_i = 0 \quad \Leftrightarrow \quad \Box_{(g)} f_i = 0 \tag{1}$$

for matter fields and

$$\begin{aligned} \partial_+\partial_-\Phi + \kappa^2\Theta &= G\langle \hat{T}_{+-}\rangle \equiv \bar{N}G\hbar\partial_+\partial_- \ln(\Phi\Theta^{-1}), \\ \Phi\partial_+\partial_- \ln\Theta &= -G\langle \hat{T}_{+-}\rangle \equiv -\bar{N}G\hbar\partial_+\partial_- \ln(\Phi\Theta^{-1}) \end{aligned} \tag{2}$$

for geometric fields $\Theta$ and $\Phi$. The terms on the right side are quantum corrections to the classical equations due to conformal anomaly and encode the backreaction of quantum radiation. As in 4D general relativity, there are constraints which are preserved by the evolution equations:

$$\begin{aligned} -\partial_-^2\Phi + \partial_-\Phi\partial_- \ln\Theta &= G\langle \hat{T}_{--}\rangle, \\ -\partial_+^2\Phi + \partial_+\Phi\partial_+ \ln\Theta &= G\langle \hat{T}_{++}\rangle. \end{aligned} \tag{3}$$

Here, $\bar{N} := N/24$, and $\langle \hat{T}_{ab}\rangle$ denotes the expectation value of the stress-energy tensor of the $N$ fields $f_i$.

We solve this system of equations as follows. As is usual, we assume that prior to $z^+ = 0$ the space-time is given by the classical vacuum solution and matter falls in from $\mathcal{I}_R^-$ after that (see Fig. 1). Therefore, to specify consistent initial data, it suffices to choose a matter profile $f_+(z^+)$ on $\mathcal{I}_R^-$ and solve for the initial $(\Theta, \Phi)$ by using (3). We then evolve $(\Theta, \Phi)$ to the future by using (2). Trivially, $f_i(z^+, z^-) = f_+(z^+)$ from (1).

 

We now discuss the interpretation of solutions via horizons, singularities, and the Bondi mass. Note first that, in analogous four-dimensional (4D) spherically symmetric reductions, $\Phi$ is related to the radius $r$ by $\Phi = \kappa^2 r^2$ [2,5]. Therefore, a point in the CGHS space-time $(M, g)$ is said to be *future marginally trapped* if $\partial_+\Phi$ vanishes and $\partial_-\Phi$ is negative there [2,8]. The quantum corrected "area" of a trapped point is given by $\mathbf{a} := (\Phi - 2\bar{N}G\hbar)$. The worldline of these marginally trapped points forms a generalized dynamical horizon (GDH). As time evolves, this area *shrinks* because of quantum radiation and finally goes to zero. At this point, it meets the spacelike singularity $\Phi = 2\bar{N}G\hbar$. The "last ray"—the null geodesic from this point to $\mathcal{I}_R^+$—is the future Cauchy horizon of the semiclassical space-time. See Fig. 1.

We assume (and this is borne out by the simulations) that the semiclassical space-time is asymptotically flat at $\mathcal{I}_R^+$ in the sense that, as $z^+ \to \infty$, the field $\Phi$ has the following behavior along $z^- = \text{const}$ lines:

$$\Phi = A(z^-)e^{\kappa z^+} + B(z^-) + O(e^{-\kappa z^+}), \tag{4}$$

where $A$ and $B$ are smooth functions of $z^-$. A similar expansion holds for $\Theta$. The physical semiclassical metric $g_{ab}$ admits an *asymptotic* time translation $t^a$. Its affine parameter $y^-$ is given by $e^{-\kappa y^-} = A(z^-)$. Up to an additive constant, $y^-$ serves as the unique physical time parameter at $\mathcal{I}_R^+$. The MFA equations imply that there is a balance law at $\mathcal{I}_R^+$ [4,7], motivating new definitions of a Bondi mass $M_{\text{Bondi}}^{\text{ATV}}$ and a manifestly positive energy flux $F^{\text{ATV}}$:

$$M_{\text{Bondi}}^{\text{ATV}} = \frac{dB}{dy^-} + \kappa B + \bar{N}\hbar G\left[\frac{d^2y^-}{dz^{-2}}\left(\frac{dy^-}{dz^-}\right)^{-2}\right], \tag{5}$$

$$F^{\text{ATV}} = \frac{\bar{N}\hbar G}{2}\left[\frac{d^2y^-}{dz^{-2}}\left(\frac{dy^-}{dz^-}\right)^{-2}\right]^2, \tag{6}$$

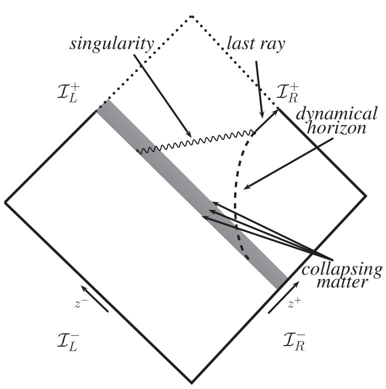


FIG. 1. A Penrose diagram of an evaporating CGHS black hole in the MFA. The incoming state is the vacuum on $\mathcal{I}_L^-$, and left-moving matter distribution on $\mathcal{I}_R^-$. The collapse creates a GDH, which subsequently evaporates. Quantum radiation fills the space-time to the causal future of matter. Inside the GDH, a singularity forms in the geometry. It meets the GDH when the latter shrinks to zero area. The last ray emanating from this meeting point is a future Cauchy horizon.

so that $d(M_{\text{Bondi}}^{\text{ATV}})/dy^- = -F^{\text{ATV}}$. In the classical theory ($\hbar = 0$), there is no energy flux at $\mathcal{I}_R^+$, and $M_{\text{Bondi}}^{\text{ATV}}$ reduces to the standard Bondi mass formula, which includes only the first two terms in (5). Previous literature [1,2,8–10] on the CGHS model used this classical expression also in the semiclassical theory. But we will see that this traditionally used Bondi mass $M_{\text{Bondi}}^{\text{Trad}}$ is physically unsatisfactory.

*Scaling and the Planck regime.*—It turns out that the mean-field theory admits a scaling symmetry. Given any solution $(\Theta, \Phi, N, f_+)$ to all the field equations and a positive number $\lambda$, $(\lambda\Theta, \lambda\Phi, \lambda N, f_+)$ is also a solution [5,11]. Under this transformation, we have

$$g^{ab} \to g^{ab}, \qquad (M, F^{\text{ATV}}, \mathbf{a}_{\text{GDH}}) \to \lambda(M, F^{\text{ATV}}, \mathbf{a}_{\text{GDH}}),$$

where $\mathbf{a}_{\text{GDH}}$ denotes the area of the GDH and $M$ is either the Bondi mass $M_{\text{Bondi}}^{\text{ATV}}$ or the ADM mass $M_{\text{ADM}}$. This symmetry implies that, physically, only the ratio $M/N$ matters. Thus, whether a black hole is "macroscopic" or "Planck size" depends on the ratios $M/N$ and $\mathbf{a}_{\text{GDH}}/N$ rather than on the values of $M$ or $\mathbf{a}_{\text{GDH}}$ themselves. Hence we are led to define

$$(M^\star, M_{\text{Bondi}}^\star, F^\star) = (M_{\text{ADM}}, M_{\text{Bondi}}^{\text{ATV}}, F^{\text{ATV}})/\bar{N} \quad \text{and} \quad m^\star = M_{\text{Bondi}}^\star|_{\text{last ray}}. \tag{7}$$

To compare these quantities to the Planck scale, note that there are subtleties as $G\hbar$ is dimensionless in 2D; careful considerations lead us to set $M_{\text{Pl}}^2 = \hbar\kappa^2/G$ and $\tau_{\text{Pl}}^2 = G\hbar/\kappa^2$ [5]. We can regard a black hole as macroscopic if its evaporation time is much larger than the Planck time. Since the energy flux is given by $F_{\text{Haw}} = (\bar{N}\hbar\kappa^2/2)$ in the external field approximation, this condition leads us to say that a black hole is macroscopic if $M^\star \gg G\hbar M_{\text{Pl}}$. Note that the relevant quantity is $M^\star$ rather than $M$. The precise nature of this scaling property was not appreciated until recently. For example, in Ref. [12] it was noted that $N$ could be "scaled out" of the problem and that the results are "qualitatively independent of $N$," whereas in fact for a given $M$ they can vary significantly as $N$ changes. Similarly, the condition that a macroscopic black hole should have large $M/N$ appears in Ref. [9]. But it was arrived at by physical considerations involving static solutions rather than an exact scaling property of the full equations.

*Results.*—Here we describe some key results from numerical solution of the CGHS equations (1) and (2). We consider two families of initial data, most conveniently described in a "Kruskal-like" coordinate $\kappa x^+ = e^{\kappa z^+}$. The first is a collapsing shell used extensively in the CGHS literature:

$$(\partial f_+/\partial x^+)^2 = \frac{M^\star}{12}\,\delta(x^+ - 1/\kappa), \tag{8}$$

parameterized by $M^\star$. The other is a smooth [$f_+(x^+)$ is $\mathcal{C}^4$], two-parameter ($\tilde{M}^\star$ and $w$) profile defined by

$$\int_0^{x^+} d\bar{x}^+ \left(\frac{\partial f_+}{\partial \bar{x}^+}\right)^2 = \frac{\tilde{M}^\star}{12}(1 - e^{(\kappa x^+ - 1)^2/w^2})^4 \theta(x^+ - 1/\kappa), \tag{9}$$

where $\theta$ is the unit step function, $w$ characterizes the width of the matter distribution, and $\tilde{M}^\star$ is related to the ADM mass via $M^\star \approx \tilde{M}^\star(1 + 1.39w)$. Unraveling of the unforeseen behavior required high precision numerics [6], which is crucial in the macroscopic mass limit that is of primary importance. Numerical solutions from both classes of initial data were obtained for a range of masses $M^\star$ from $2^{-10}$ to 16, a range of widths from $w = 0$ to $w = 4$, and $\bar{N}$ varying from 0.5 to 1000. Since we are interested in initially macroscopic black holes, here we will focus on $M^\star \geq 1$ and, since the computations did bear out the scaling behavior, on $\bar{N} = 1$. We set $\hbar = G = \kappa = 1$.

Our numerical simulations show that, as expected, the semiclassical space-time is asymptotically flat at $\mathcal{I}_R^+$, but, in contrast to the classical theory, $\mathcal{I}_R^+$ is incomplete; i.e., $y^-$ has a finite value at the last ray. However, the dynamics also exhibits some surprising features.

First, the traditionally used Bondi mass $M_{\text{Bondi}}^{\text{Trad}}$ can become negative and large even when the GDH is macroscopic. For CGHS black holes, negative $M_{\text{Bondi}}^{\text{Trad}}$ was known to occur [13] but only for black holes which are of Planck size even before evaporation begins. For initially macroscopic black holes, the standard paradigm assumed that $M_{\text{Bondi}}^{\text{Trad}}$ is positive and tends to zero as the GDH shrinks (so that one can attach a "flat corner" of Minkowski space to the future of the last ray). Second, while the improved Bondi mass $M_{\text{Bondi}}^{\text{ATV}}$ does remain positive throughout evolution, at the last ray it can be large. In fact this "end state" exhibits a universality shown in Fig. 2, where $m^\star$, the final value of $M_{\text{Bondi}}^\star$, is plotted against the rescaled ADM mass $M^\star$ for a range of initial data. It is clear from the plot that there is a qualitative difference between $M^\star \gtrsim 4$ and $M^\star \lesssim 4$. In the first case the value of the end-point Bondi mass is universal: $m^\star \approx 0.864$. For $M^\star < 4$, on the other hand, the value of $m^\star$ depends sensitively on $M^\star$. Thus in the MFA it is natural to regard CGHS black holes with $M^\star \gtrsim 4$ as *macroscopic* and those with $M^\star \lesssim 4$ as *microscopic*. Past numerical studies [3,10,12,13] missed the universal behavior mainly because they investigated only *microscopic* cases ($M^\star \leq 2.5$ in all prior studies).

Third, for macroscopic ($M^\star \gtrsim 4$) black holes that form *promptly*, after early transient behavior, dynamics of physical quantities at the GDH and at $\mathcal{I}_R^+$ approach *universal curves.* By promptly, we mean the characteristic width of the ingoing pulse is less than that of the initial GDH (more precisely, $w/M^\star \lesssim 0.1$). This is most clearly demonstrated in the behavior of the flux $F^\star$, or equivalently the Bondi mass $M_{\text{Bondi}}^\star$, measured at $\mathcal{I}_R^+$. An appropriately shifted affine parameter $y_{\text{sh}}^- = y^- + \text{const}$ provides an invariantly defined time coordinate, and Fig. 3 shows the universality of evolution of $F^\star$ and $M_{\text{Bondi}}^\star$ with respect to it. The shift aligns the $y^-$ coordinates among the solutions, which we are free to do as $y^-$ is only uniquely defined to within a (physically irrelevant) additive constant. Finally, note that this universality is qualitatively different from the known uniqueness results for solutions of certain simplified soluble models [14]. It occurs only if the black hole is initially macroscopic, formed by a prompt collapse. And in this case, after the transient phase, the behavior of physical quantities at $\mathcal{I}_R^+$ does not even depend on the mass.

The situation with universality bares parallels to the discovery of critical phenomena at the threshold of gravitational collapse in classical general relativity [15] where universal properties were discovered in a system that, at the time, seemed to have been already explored exhaustively. Of course, numerical investigations cannot *prove* universality; here we studied only two families of initial data. However, since these families, in particular, the distribution, are not "special" in any way, we believe this is strong evidence that universality is a feature of the "pure" quantum decay of a GDH, pure in that the decay is not contaminated by a continued infall from $\mathcal{I}_R^-$.

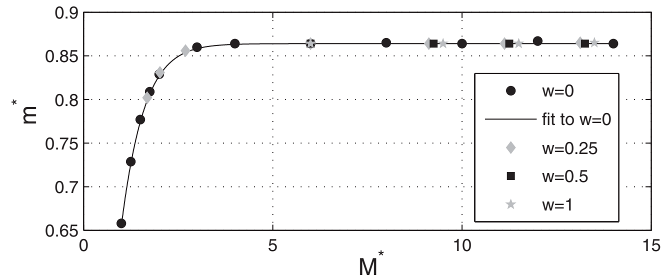


FIG. 2. The final mass $m^\star$ versus the initial mass $M^\star$ (7) for a variety of initial data (8) and (9). The curve fit to the data is $m^\star = \alpha(1 - e^{-\beta(M^\star)^\gamma})$, with $\alpha \approx 0.864$, $\beta \approx 1.42$, and $\gamma \approx 1.15$.

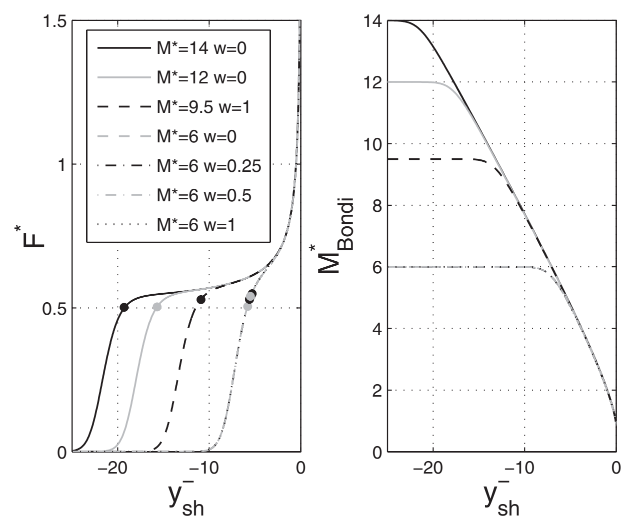


FIG. 3. $F^\star$ and $M_{\text{Bondi}}^\star$ of Eq. (7) plotted against $y_{\text{sh}}^-$ for solutions with several values of parameters $M^\star$ and $w$ of Eqs. (8) and (9). In all cases, $F^\star$ starts at 0 in the distant past ($\kappa y_{\text{sh}}^- \ll -1$) and then joins a universal curve at a time that depends on the initial mass. The time when the dynamical horizon first forms is marked on each flux curve.

Finally, along the last ray, our simulations show that curvature remains finite. Thus, contrary to wide spread belief, based in part on Ref. [3], and in contrast to simplified and soluble models, there is no ''thunderbolt singularity'' in the metric (for details, see [5]).

*Conclusions.—*In the external field approximation, the energy flux is initially zero and, after the transient phase, quickly asymptotes to the Hawking value $F_{\rm Haw} = \bar{N}\hbar\kappa^2/2 \equiv 0.5$ for the constants used in the simulations shown here. In the MFA calculation, on the other hand, at the end of the transient phase the energy flux is *higher* than this value, keeps monotonically increasing, and is about 70% greater than $F_{\rm Haw}$ when $M_{\rm Bondi} \sim 2\bar{N}M_{\rm Pl}$ (see Fig. 3). One might first think that the increase is because, as in 4D, the black hole gets hotter as it evaporates. This is *not* so: For CGHS black holes, $T_{\rm Haw} = \kappa\hbar/2\pi$ and $\kappa$ is an absolute constant. Rather, the departure from $F_{\rm Haw} = 0.5$ shows that, once the backreaction is included, the flux fails to be thermal at the late stage of evaporation, *even while the black hole is macroscopic*. This removes a widely quoted obstacle against the possibility that the outgoing quantum state is pure in the full theory.

In the classical solution, $\mathcal{I}_R^+$ is *complete* and its causal past covers only a part of space-time; there is an event horizon. But $\mathcal{I}_R^+$ is smaller than $\mathcal{I}_L^-$ in a precise sense: $z^-$, the affine parameter along $\mathcal{I}_L^-$, is finite at the future end of $\mathcal{I}_R^+$. This is why pure states on $\mathcal{I}_L^-$ of a *test* quantum field $\hat{f}_-$ on the classical solution evolve to mixed states on $\mathcal{I}_R^+$ [4,7], i.e., why the $S$ matrix is nonunitary. In the MFA, by contrast, our analysis shows that as expected $y^-$ is *finite* at the last ray on $\mathcal{I}_R^+$. Thus, $\mathcal{I}_R^+$ is incomplete whence we cannot even ask if the semiclassical space-time admits an event horizon; what forms and evaporates is, rather, the GDH. However, this incompleteness also opens the possibility that $\bar{\mathcal{I}}_R^+$, the right null infinity of the full quantum space-time, may be larger than $\mathcal{I}_R^+$ and unitarity may be restored. Indeed, since there is no thunderbolt, space-time can be continued beyond the last ray. In the mean-field theory, the extension is ambiguous. But it is reasonable to expect that the ambiguities will be removed by full quantum gravity [16]. Indeed, since we have only $(0.864/24)M_{\rm Pl}$ of Bondi mass left over at the last ray *per evaporation channel* (i.e., per scalar field), it is reasonable to assume that this remainder will quickly evaporate after the last ray and $M_{\rm Bondi}^{\rm ATV}$ and $F^{\rm ATV}$ will continue to be zero along the quantum extension $\bar{\mathcal{I}}_R^+$ of $\mathcal{I}_R^+$. The form of $F^{\rm ATV}$ now implies that $\bar{\mathcal{I}}_R^+$ is ''as long as'' $\mathcal{I}_L^-$ and hence the $S$ matrix is unitary: The vacuum state on $\mathcal{I}_L^-$ evolves to a many-particle state with a *finite* norm on $\bar{\mathcal{I}}_R^+$ [4,7]. Thus unitarity of the $S$ matrix follows from rather mild assumptions on what transpires beyond the last ray.

Note, however, this unitarity of the $S$ matrix from $\mathcal{I}_R^-$ to the extended $\mathcal{I}_R^+$ does *not* imply that all the information in the infalling matter on $\mathcal{I}_R^-$ is imprinted in the outgoing state on $\bar{\mathcal{I}}_R^+$. Indeed, the outgoing quantum state is completely determined by the function $y^-(z^-)$, and our universality results imply that, on $\mathcal{I}_R^+$, this function depends only on $M_{\rm ADM}$ and not on further details of the matter profile [5]. Since only a tiny fraction of Planck mass is radiated per channel in the portion of $\bar{\mathcal{I}}_R^+$ that is not already in $\mathcal{I}_R^+$, it seems highly unlikely that the remaining information can be encoded in the functional form of $y^-(z^-)$ in that portion. Thus, information in the matter profile on $\mathcal{I}_R^-$ will not all be recovered at $\bar{\mathcal{I}}_R^+$ even in the full quantum theory of the CGHS model. This contradicts a general belief; indeed, because the importance of $y^-(z^-)$ was not appreciated and its universality was not even suspected, there have been attempts at constructing mechanisms for recovery of this information [9].

In summary, in 2D there are two distinct issues: (i) unitarity of the $S$ matrix from $\mathcal{I}_L^-$ to $\bar{\mathcal{I}}_R^+$ and (ii) recovery of the infalling information on $\mathcal{I}_R^-$ at $\bar{\mathcal{I}}_R^+$. The distinction arises because right and left pieces of $\mathcal{I}^\pm$ do not talk to each other. In 4D, by contrast, we have only one $\mathcal{I}^-$ and only one $\mathcal{I}^+$. Therefore if the $S$ matrix from $\mathcal{I}^-$ to $\mathcal{I}^+$ is unitary, all information in the ingoing state at $\mathcal{I}^-$ is automatically recovered in the outgoing state at $\mathcal{I}^+$. To the extent that the CGHS analysis provides guidance for the 4D case, it suggests that unitarity of the $S$ matrix should continue to hold also in 4D [7].

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