Torsional Monopoles and Torqued Geometries in Gravity and Condensed Matter

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Torsional degrees of freedom play an important role in modern gravity theories as well as in condensed matter systems where they can be modeled by defects in solids. Here we isolate a class of torsion models that support torsion configurations with a localized, conserved charge that adopts integer values. The charge is topological in nature, and the torsional configurations can be thought of as torsional ''monopole'' solutions. We explore some of the properties of these configurations in gravity models with a nonvanishing curvature and discuss the possible existence of such monopoles in condensed matter systems. To conclude, we show how the monopoles can be thought of as a natural generalization of the Cartan spiral staircase.

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The analogy between geometry and defects in gravity theories and in theories of elasticity in solids is an old and well-developed field of study [1–7]. Disclinations and dislocations in crystals are defects in the ordered lattice which carry finite curvature and torsion, respectively. Transporting a particle around a disclination (dislocation) produces a nonzero rotation (translation) by the end of the cycle. Dislocations are particularly interesting because, while sources of curvature are ubiquitous in the natural Universe, the effects of torsion are less pronounced experimentally [8,9]. In solids, however, dislocations affect many material properties and are present even in the cleanest materials. Thus condensed matter systems can provide useful laboratories to study torsion.

The defects that we will describe cannot be described by the classical geometric theory of elasticity. Instead, these defects may occur in materials described by micropolar elasticity theory [10]. Micropolar elasticity theory (or Cosserat elasticity) is a simple extension of classical elasticity to include local orientational degrees of freedom of the constituent particles or molecules of the elastic medium. The defect we investigate, which we dub a torsional ''monopole'' (TM), does not require a lattice deformation but a deformation texture in the local rotational degrees of freedom. Such defects could exist in biological or granular systems (two common systems described by micropolar elasticity) and may affect solids with a strong coupling between orbital electronic motion and local spin or orbital degrees of freedom. The structure of a TM is shown in Fig. 1, and we will present a general treatment of these defects in a gravitational context in flat and curved space and give an explicit construction of a TM while showing its relation to defects in solids and the ''Cartan spiral staircase.'' It is important to note that these defects lie outside the typical topological defects found in gauge theories in, e.g., Refs. [11,12]. We begin with a simple formulation of our construction in flat space. To isolate the purely torsional degrees of freedom, we will begin with three highly constraining *Ansätze*. In order to focus on the minimal, kinematical properties of torsional defects, and to make the theory as generalizable as possible to various condensed matter and gravity theories, we will not assume an underlying dynamical geometric theory. We begin with the typical ingredients of Einstein-Cartan gravity, but we will not impose the Einstein-Cartan or any other dynamical equations of motion. For generality, we will work with the 4D Lorentzian theory, but all of the major results are applicable to 3D Euclidean systems. Thus, we take

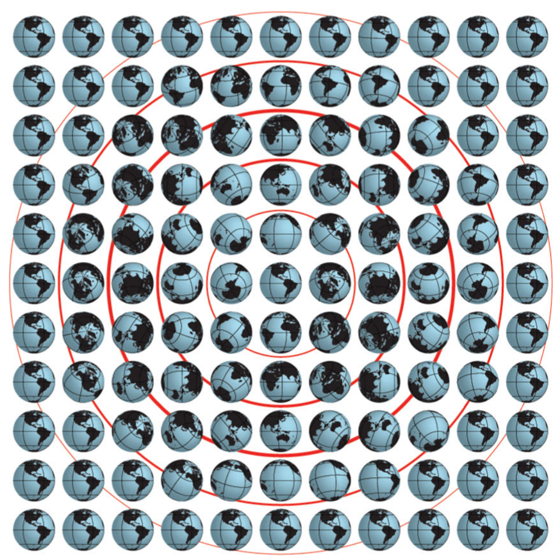

FIG. 1 (color online). A cross section through the origin of a torsion monopole with $Q = 1$. The globe picks out three directions which form an orthonormal triad. The monopole pictured has a radius of five lattice sites, and it is formed by rotating each globe along a radial line directed from the origin by an amount proportional to the radius of the lattice site, until the angle of rotation reaches 2π at $r = 5$.

the geometry to be described by a $\mathrm{Spin}(3,1) \simeq \overline{SO}(3,1) \simeq SL(2,\mathbb{C})$ gauge theory, whose connection coefficient in a local trivialization is the spin connection ω, and a (co-) frame (tetrad) field e, which is a one-form taking values in the adjoint representation of Spin(3, 1). The three *Ansätze* we will make are (i) the manifold has the topology of $\mathbb{R}^4$ and the metric induced from the coframe e^I is the flat Minkowski metric, i.e., in Cartesian coordinates $\boldsymbol{g} = \eta_{IJ} e^I \otimes e^J = -dt^2 + dx^2 + dy^2 + dz^2$; (ii) the spin connection ω^{IJ} has a vanishing curvature and, thus, we have a Weitzenböck spacetime; and (iii) the spin connection asymptotically approaches the Levi-Civita connection (the special connection compatible with e^I) at spatial infinity so that $\lim_{r\to\infty}\omega = \Gamma[e]$. We refer to a geometry satisfying these conditions as a *torqued geometry* for reasons that will become clear.

We will denote the Levi-Civita connection associated with the tetrad by $\Gamma = \Gamma[e]$, i.e., $D_\Gamma e^I = de^I + \Gamma^I{}_J \wedge e^J = 0$. For any connection A let the curvature $R_A = dA + A \wedge A$. Since the metric is Minkowski, $R_\Gamma = 0$, and since the space is Weitzenböck, $R_\omega = 0$ as well. However, we do not assume that the spin connection is *compatible* with the tetrad, and the (generically nonzero) torsion is given by the usual expression $T^I = D_\omega e^I = de^I + \omega^I{}_J \wedge e^J$. The assumption that the spin connection asymptotically approaches the Levi-Civita connection at spatial infinity implies that $\lim_{r\to\infty} T^I = 0$. The *rate* at which the torsion must tend to zero will be fixed later. It will be useful to express the spin connection ω in terms of the Levi-Civita connection Γ by

$$\omega^I{}_J = \Gamma^I{}_J + C^I{}_J, \tag{1}$$

where $C^{IJ} = C^{[IJ]}{}_\mu dx^\mu$ is the contorsion tensor [13].

In Minkowski space all flat connections are gauge related [since $\pi_1(\mathbb{R}^4) = 0$], and, thus, we can express the spin connection in terms of the Levi-Civita connection as (in the fundamental representation, where we drop explicit indices)

$$\omega = g\Gamma g^{-1} - dgg^{-1}. \tag{2}$$

The group element $g = g(x)$ is the "relative gauge" between the spin connection and the tetrad. The term torqued geometry is in reference to the relationship between the spin connection and Levi-Civita connection in Eq. (2). Using Eqs. (1) and (2) we have

$$C = g\Gamma g^{-1} - dgg^{-1} - \Gamma = -D_\Gamma g g^{-1}. \tag{3}$$

We will occasionally use the "trivial" gauge where the tetrad in Cartesian coordinates is $\overset{0}{e}{}^I = \delta^I_\mu dx^\mu$ and the corresponding Levi-Civita connection $\Gamma[\overset{0}{e}] = 0$. For $\overset{0}{e} = heh^{-1}$ the spin connection is

$$\omega' = h\omega h^{-1} - dhh^{-1} = -dg' g'^{-1} \quad \text{with } g' = hgh^{-1}. \tag{4}$$

One of the key properties of the torsional configurations that follows from our flatness *Ansatz* is the existence of a conserved current. To see this, consider the curvature of the spin connection expressed in terms of the contorsion tensor. From the definition we have

$$R_\omega = R_\Gamma + D_\Gamma C + C \wedge C = D_\Gamma C + C \wedge C = 0,$$

which follows since both R_Γ and $R_\omega = 0$. Now consider

$$\begin{aligned}\Omega &\equiv \mathrm{Tr}_D\Big(\frac{1}{4\pi^2}(D_\Gamma C + C\wedge C)\wedge(D_\Gamma C + C \wedge C)\Big)\\ &= \frac{1}{4\pi^2} d\,\mathrm{Tr}_D\Big(C \wedge D_\Gamma C + \frac{2}{3} C\wedge C\wedge C\Big)\end{aligned} \tag{5}$$

with $\mathrm{Tr}_D(\cdot) = \frac{1}{D}\mathrm{Tr}(\cdot)$, where D is the dimension of the representation. Defining the topological current three-form $\xi^{(3)}_C \equiv \frac{1}{4\pi^2}\mathrm{Tr}_D(C\wedge D_\Gamma C + \frac{2}{3} C\wedge C\wedge C)$ and recognizing that $\Omega = 0$, we see that the current is conserved: $d\xi^{(3)}_C = 0$.

The current allows us to define a conserved charge. To do this, we will first fix our asymptotic boundary conditions on the torsion so that the defects we will consider are spatially isolated. It is sufficient to assume that in the trivial gauge C, and thus T, fall off like $\frac{1}{r}$ as $r \to \infty$. Thus, in this gauge the relative gauge given in Eq. (4) must be such that $g' \to$ const near spatial infinity. This allows for the standard compactification $\Sigma \simeq \mathbb{R}^3 \cup \{\infty\} \simeq \mathbb{S}^3$ for a spatial slice Σ. Because of the assumed falloff conditions on the torsion, the flux of current through the timelike cylinder at asymptotic infinity is zero, so we define the *conserved* torsional charge

$$Q \equiv \int_\Sigma \xi^{(3)}_C = -\frac{1}{12\pi^2}\int_\Sigma \mathrm{Tr}_D(C\wedge C\wedge C). \tag{6}$$

One can also define a conserved dual current by taking the internal Hodge dual of one of the components in the four-form of (5); however, the corresponding charge vanishes identically for our class of geometries.

The conservation of the charge is a purely kinematic property, *independent* of any dynamics to which the TMs are subjected. Indeed, the charges are *topological* in nature, and, under small deformations $\{\boldsymbol{\delta} e, \boldsymbol{\delta}\omega\}$ that preserve the flatness constraints, we have $\boldsymbol{\delta} Q = 0$. In fact, Q takes quantized integer values as we will now show.

First, we note that, despite its similarity to the Chern-Simons functional, the charge Q is different in that it is identically gauge *invariant* under both large and small gauge transformations. Thus, we can choose a convenient gauge in order to compute the charge. We choose the trivial gauge where the contorsion is given by $C = -dg'g'^{-1}$ [cf. Eq. (4)]. Thus in this gauge using Eq. (6), we have [using the shorthand notation $(dgg^{-1})^3 = \mathrm{Tr}_D(dgg^{-1}\wedge dgg^{-1}\wedge dgg^{-1})$]

$$Q = \frac{1}{12\pi^2}\int_\Sigma (dg'g'^{-1})^3 = \frac{1}{12\pi^2}\int_\Sigma (dgg^{-1})^3. \tag{7}$$

We recognize the last line as the index or winding number of the map $g:\Sigma \to \mathrm{Spin}(3,1)$. The winding number is well defined since Spin(3, 1) has $SU(2)$ as its maximal compact subgroup. Such maps are classified by $\pi_3[\mathrm{Spin}(3,1)] = \mathbb{Z}$ and thus $Q \in \mathbb{Z}$.

To construct explicit configurations with nonzero charge, we borrow from well-known results in $SU(2)$ Yang-Mills theories (see, e.g., [14]). We will work in the trivial gauge in Cartesian coordinates so the tetrad is $e^I_\mu = \delta^I_\mu$ and $\Gamma^{IJ} = 0$. In this gauge $\omega = -dg' g'^{-1}$, and defining $\tau_i = \frac{1}{2}\epsilon_{ijk}\gamma^j\gamma^k$, where γ^I are generators of the Clifford algebra and $i, j, k = 1, 2, 3, \ldots$ are spatial indices, we take g' to be [(i) labels which TM]

$$g' = g_{(i)} = \cos(\chi_{(i)})\mathbf{1} + \sin(\chi_{(i)})\frac{x^a - x^a_{(i)}}{|\vec{x} - \vec{x}_{(i)}|}\tau_a, \qquad (8)$$

where $\chi_{(i)} = \chi_{(i)}(\Delta r_{(i)})$ with $\Delta r_{(i)} = |\vec{x} - \vec{x}_{(i)}|$ is any continuous and differentiable function that monotonically increases from 0 at $\Delta r_{(i)} = 0$ to π at $\Delta r_{(i)} = \infty$. To ensure the configuration is well behaved, we assume $\frac{\partial \chi_{(i)}}{\partial \Delta r_{(i)}}|_{\Delta r_{(i)}=0} = \frac{\partial \chi_{(i)}}{\partial \Delta r_{(i)}}|_{\Delta r_{(i)}=\infty} = 0$. For a TM of charge q located at the origin ($\vec{x}_{(i)} = 0$), the contorsion is

$$C^{ij} = -2[\epsilon^{ij}{}_k \hat{X}^k d\chi + \sin(\chi)\cos(\chi)\epsilon^{ij}{}_k d\hat{X}^k - 2\sin^2(\chi)\hat{X}^{[i} d\hat{X}^{j]}]. \qquad (9)$$

The torsional charge for this configuration can be explicitly computed to yield $Q = 1$. We can then use this group element to generate multiple TM solutions of the generic form $C = -dg' g'^{-1}$ with $g' = g^{q_1}_{(1)} g^{q_2}_{(2)} \ldots g^{q_N}_{(N)}$ and charge $Q = q_1 + q_2 + \cdots + q_N$.

It is worthwhile to address a potential source of confusion stemming from the analogous geometric constructs in Yang-Mills theory. We have referred to the configurations above as monopoles because they are spatially isolated torsional defects of a topological nature. However, typical nomenclature in Yang-Mills theories associates monopoles with $\pi_2(G)$ and instantons with $\pi_3(G)$. Despite similarities to analogous structures in Yang-Mills theories, the TM has some fundamental differences. The key property that allows for a stable, gauge invariant topological structure in three dimensions is that the topological charge can be identified not with a single Chern-Simons functional but with the difference of two Chern-Simons functionals: $Q = \int_\Sigma [CS(\omega) - CS(\Gamma)]$. The resulting quantity is invariant under large gauge transformations, unlike either of its two constituents, but the quantity picks out the winding number of the relative gauge between the two connections.

Our major results can be extended to a class of curved spacetimes. Given a tetrad e and its associated $\Gamma[e]$, we focus on the class of geometries in which the spin connection differs from $\Gamma[e]$ only by a relative gauge:

$$\omega = g\Gamma g^{-1} - dgg^{-1}, \qquad R_\omega = gR_\Gamma g^{-1}. \qquad (10)$$

The contorsion is still given by Eq. (3), but the curvature is nonvanishing so generically the contorsion satisfies the condition $D_\Gamma C + C \wedge C = gR_\Gamma g^{-1} - R_\Gamma$. Nevertheless, there is still a conserved current since

$$\Omega = \frac{1}{4\pi^2}\mathrm{Tr}_D(R_\omega \wedge R_\omega - R_\Gamma \wedge R_\Gamma) = 0 \qquad (11)$$

and $\Omega = d[CS(\omega) - CS(\Gamma)] \equiv d\tilde{J} = 0$. The current three-form is entirely torsional in nature as it can be written

$$\tilde{J} = \xi^{(3)}_C + \frac{1}{2\pi^2}\mathrm{Tr}_D(2C \wedge R_\Gamma). \qquad (12)$$

This current is conserved, and being the difference of two Chern-Simons functionals that differ by a relative gauge g, we clearly have $Q = \frac{1}{12\pi^2}\int_{\mathbb{S}^3}(dgg^{-1})^3$. Thus, the charge is topologically quantized.

Now we return to flat space to discuss the analogy with defects in solids. Thus far, we have viewed torqued geometries as a deformation of the spin connection by a relative gauge transformation. To model a TM, it is convenient to make a (true, not relative) gauge transformation to absorb the deformation entirely in the tetrad. It is sufficient to work with 3D Euclidean space, and we denote the triad by E^i_a. The geometric variables describing a torqued geometry before the transformation are $E = \overset{0}{E}$, and $\omega = g\overset{0}{\Gamma} g^{-1} - dgg^{-1}$, where $\overset{0}{E}$ is a fiducial Euclidean flat tetrad and $\overset{0}{\Gamma} = \Gamma[\overset{0}{E}]$ is the corresponding Levi-Civita connection. Gauge transforming by g^{-1}, we obtain $E' = g^{-1}\overset{0}{E}_g$ and $\omega' = g^{-1}\omega g - dg^{-1}g = \overset{0}{\Gamma}$. Now the deformation induced by the relative gauge is encoded in the triad as opposed to the spin connection. For a single TM of charge q located at the origin, the relative gauge $g = \cos[q\chi(r)]\mathbf{1} + i\sin[q\chi(r)]\hat{x}_a \overset{0}{E}{}^a_i \sigma^i$, where σ^i are the Pauli matrices, is an element of $SU(2)$ at each point. For clarity, fix the fiducial triad to be $\overset{0}{E}{}^i_a = \delta^i_a$. We focus on the behavior of E'^i_a along any ray beginning at the origin and ending at asymptotic infinity. The gauge transformation represents a rotation at each point around the axis defined by the ray. Since $\chi(0) = 0$ at the origin and monotonically increases to $\chi(\infty) = \pi$, the relative gauge represents a spatial rotation of the fiducial triad around the radial axis such that the rotation is the identity at the origin and increases monotonically to $2\pi q$ at infinity. The direction is clockwise or counterclockwise depending on the sign of q.

The realization of a model for the TM in a condensed matter system points out some of the deficiencies of the classical geometric theory of elasticity. The classical geometric theory of dislocations and disclinations has the Euclidean Poincaré group $G = SO(3) \ltimes T(3)$ as a gauge group, where $T(3)$ is the set of three-dimensional translations [15]. Disclinations are defects associated with the rotational degrees of freedom, and in this model, the absence of disclinations is associated with the vanishing of the curvature of the spin connection $R_\omega = 0$ [2–6]. Thus, the TM, which can exist in geometries with a vanishing curvature, is not composed of disclinations. Next we consider pure dislocations which are described by a nonzero torsion associated with the translation group. In a solid we have a lattice which breaks the continuous translation symmetry down to a discrete subgroup. The topological charges of defects in the translation sector are thus given by $\pi_n[T(3)/\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}] = \pi_n(T^3)$, where π_n is the nth homotopy group, $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ represents the space of 3D discrete translations, and T^3 is the 3-torus. The only topologically stable defects are line defects [dislocations due to $\pi_1(T^3) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$], and thus our *pointlike* TM is not a dislocation. In fact, these arguments are immediately apparent if we choose the gauge for the TM where the entire deformation is in the triad. This deformation does not require a lattice deformation and thus is not effectively captured in the classical elasticity theory.

To support the TM, we need to consider the local rotational degrees of freedom of the objects forming the elastic medium and thus materials described by micropolar elasticity [5,10]. We imagine molecules or grains to which a local triad is associated. This set of axes describes the local orientation of the molecule. A fiducial geometry has all the local triads aligned, and the TM is a defect texture centered at the origin (see Fig. 1). The molecules along a radial ray are rotated around the ray axis. If the orientations are realigned outside a radius R, then the topological charge is equal to the number of revolutions carried out between $r = 0$ and $r = R$. In the continuum theory, the gauge group (in the 3D Euclidean case) is $G = SU(2)$, and the isometry group is the subgroup that leaves the triad fixed, namely, the rotations by 2π forming a $\mathbb{Z}_2$ subgroup. Thus, the relevant homotopy group is $\pi_3[SU(2)/\mathbb{Z}_2] = \pi_3[SO(3)]$. The existence of the TM is a reflection of the property $\pi_3[SO(3)] = \mathbb{Z}$.

These defects will affect the elastic behavior of micropolar media, but perhaps it is more interesting to consider possible effects in the electronic behavior of solids. Although we leave this analysis open for future work, we comment that these defects would likely affect the electronic behavior of materials with strong spin-orbit coupling since they are very sensitive to the local orientation of the orbitals. In fact, the new class of $(3 + 1)$-dimensional topological insulators [16,17] at low energy are described by the massive Dirac Hamiltonian $H = p_i\Gamma^i + m\Gamma^0$ for a trivial geometric background. When coupled to a background torsion monopole, H gains a term proportional to $\{q\partial_r\chi(r) + r^{-1}\sin[2q\chi(r)]\}\Gamma^5$. This coupling to the axial current will cause Dirac fermion states to be repulsed or attracted by a TM depending on the amount of left- or right-handed character mixed into the state. Additionally, half-integer spin particles which adiabatically pass through a TM from the origin to infinity pick up a phase of -1 from the 2π spin rotation which can lead to interference effects. Both of these effects will affect the electronic structure near a TM and may be measurable.

Using TM we can generalize Cartan's spiral staircase to the spherical case (which we might refer to as Cartan's spiral stairway to heaven). Cartan's spiral staircase is the name given to a model for a space with torsion first described in 1922 [5,7,18]. We will choose a particular $\chi(r)$ that makes the analogy as clear as possible. Suppose one were sitting on the surface of a sphere (say, the Earth) at radius r_0 and desired to build a spiral stairway analogous to Cartan's spiral staircase but oriented in the radial direction. Our model is easiest to understand by rotating the Euclidean triad and fixing the spin connection to be $\omega^{ij} = 0$. Take the triad E^I above with

$$\chi(r) = \begin{cases} 0 & \text{for } 0 \leq r \leq r_0, \\ \frac{\pi}{\lambda}(r - r_0) & \text{for } r_0 \leq r \leq r_0 + q_0\lambda, \\ \pi & \text{for } r_0 + \lambda \leq r. \end{cases} \qquad (13)$$

This gives a TM localized within radius $r_0 + q_0\lambda$ with torsional charge $Q = q_0$. By using the geometric description above, this TM is easy to visualize. Traveling from the sphere at r_0 outward to $r_0 + q_0\lambda$, the triad rotates by q_0 full turns around the radial ray. To extend the stairway, we can just increase the topological charge by sending $q_0 \to \infty$, which generates Cartan's spiral stairway to heaven.

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