Linking Quantum Discord to Entanglement in a Measurement

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We show that a von Neumann measurement on a part of a composite quantum system unavoidably creates distillable entanglement between the measurement apparatus and the system if the state has nonzero quantum discord. The minimal distillable entanglement is equal to the one-way information deficit. The quantum discord is shown to be equal to the minimal partial distillable entanglement that is the part of entanglement which is lost, when we ignore the subsystem which is not measured. We then show that any entanglement measure corresponds to some measure of quantum correlations. This powerful correspondence also yields necessary properties for quantum correlations. We generalize the results to multipartite measurements on a part of the system and on the total system.

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Quantum entanglement is by far the most famous and best studied kind of quantum correlation [1]. One reason for this situation is the fact that entanglement plays an important role in quantum computation [2]. It was even believed that entanglement is the reason why a quantum computer can perform efficiently on some problems which cannot be solved efficiently on a classical computer. The situation started to change after a computational model was presented which is referred to as "the power of one qubit" with the acronym DQC1 [3,4]. Here, using a mixed separable state allows for efficient computation of the trace of any n-qubit unitary matrix. This problem is believed to be not solvable efficiently on a classical computer [4,5]. The fact that no entanglement is present in this model was one of the main reasons why new types of quantum correlations were studied during the past few years [6–9]. One of the measures of quantum correlations, the quantum discord [6], was considered to be the figure of merit for this model of quantum computation [10].

In this Letter, we introduce an alternative approach to quantum correlations via an interpretation of a measurement. In order to perform a von Neumann measurement on a system S in the quantum state ρ^S, correlations between the system and the measurement apparatus M must be created. As a simple example we consider a von Neumann measurement in the eigenbasis $\{|i^S\rangle\}$ of the mixed state $\rho^S = \sum_i p_i |i^S\rangle\langle i^S|$ with the eigenvalues p_i. Correlations between the measurement apparatus M and the system are found in the final state of the total system $\rho_{\mathrm{final}} = \sum_i p_i |i^M\rangle\langle i^M| \otimes |i^S\rangle\langle i^S|$, where $|i^M\rangle$ are orthogonal states of the measurement apparatus M. In this state ρ_{final} the correlations between M and the system S are purely classical, and no entanglement is created. The situation changes completely if we consider partial von Neumann measurements; that is, they are restricted to a part of the system. In our main result in Theorem 1 we will show that in this case creation of entanglement is usually unavoidable. We use this result to show the close connection of our approach to the one-way information deficit [8] before we extend our ideas to the quantum discord [6] in Theorem 2 and following.

If we consider bipartite quantum states ρ^{AB}, and von Neumann measurements on A with a complete set of orthogonal rank one projectors $\Pi_i^A = |i^A\rangle\langle i^A|$, $\sum_i \Pi_i^A = \mathbb{1}_A$, then the quantum discord is defined as [6]

$$\delta^{\rightarrow}(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \min_{\{\Pi_i^A\}} \sum_i p_i S(\rho_i), \qquad (1)$$

with $p_i = \mathrm{Tr}[\Pi_i^A \rho^{AB} \Pi_i^A]$ being the probability of the outcome i, and $\rho_i = \Pi_i^A \rho^{AB} \Pi_i^A / p_i$ being the corresponding state after the measurement. The quantum discord is nonnegative and zero if and only if the state ρ^{AB} has the form $\rho^{AB} = \sum_i p_i |i^A\rangle\langle i^A| \otimes \rho_i^B$ with orthogonal states $|i^A\rangle$. Recently an interpretation of the quantum discord was found using a connection to extended state merging [11,12]. Another interpretation was given earlier in [13].

A closely related quantity is the one-way information deficit [8,14]. For a bipartite state ρ^{AB} it is defined as the minimal increase of entropy after a von Neumann measurement on A:

$$\Delta^{\rightarrow}(\rho^{AB}) = \min_{\{\Pi_i^A\}} S\left(\sum_i \Pi_i^A \rho^{AB} \Pi_i^A\right) - S(\rho^{AB}), \qquad (2)$$

where the minimum is taken over $\{\Pi_i^A\}$ as defined above Eq. (1). The one-way information deficit is non-negative and zero only on states with zero quantum discord. It can be interpreted as the amount of information in the state ρ^{AB}, which cannot be localized via a classical communication channel from A to B [14].

Given a bipartite quantum state ρ^{AB}, we recall that a partial von Neumann measurement on A can be described by coupling the system in the state ρ^{AB} to the measurement apparatus M in a pure initial state $|0^M\rangle$, $\rho_1 = |0^M\rangle\langle 0^M| \otimes \rho^{AB}$, and applying a unitary on the total state [15], $\rho_2 = U \rho_1 U^\dagger$. This situation is illustrated in Fig. 1. As we will consider only measurements on the subsystem

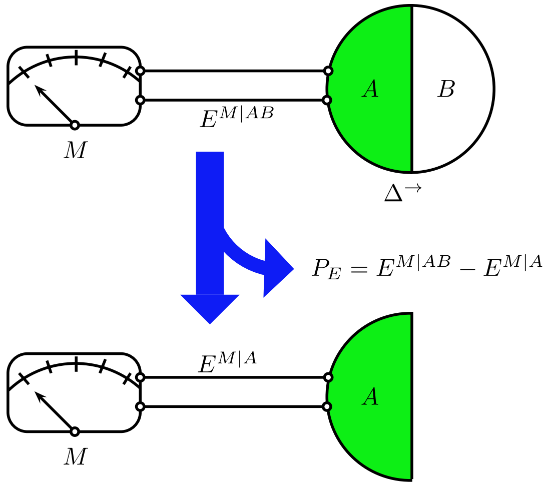

FIG. 1 (color online). A measurement apparatus M is used for a von Neumann measurement on A (green colored area), which is part of the total quantum system AB. The measurement implies a unitary evolution on the system MA, which can create entanglement $E^{M|AB}$ between the apparatus and the system. The partial entanglement $P_E = E^{M|AB} - E^{M|A}$ quantifies the part of entanglement which is lost when ignoring B.

A, the corresponding unitary U has the form $U = U_{MA} \otimes \mathbb{1}_B$. In the following, we will say that a unitary U realizes a von Neumann measurement $\{\Pi_i^A\}$ on A, if for any quantum state ρ^{AB} holds: $\mathrm{Tr}_M[U(|0^M\rangle\langle 0^M| \otimes \rho^{AB})U^\dagger] = \sum_i \Pi_i^A \rho^{AB} \Pi_i^A$. The measurement outcome is then obtained by measuring the apparatus M in its eigenbasis.

The entanglement between the apparatus M and the system AB in the state ρ_2 will be called entanglement created in the von Neumann measurement $\{\Pi_i^A\}$ on A. Given a state ρ^{AB}, we want to quantify the minimal entanglement created in a von Neumann measurement on A, minimized over all complete sets of rank one projectors $\{\Pi_i^A\}$. The minimal amount will be called E_{meas}, and it will depend on the entanglement measure used. In the following, the entanglement measure of interest will be the distillable entanglement E_D, which is defined in [16,17]. Thus, we define E_{meas} as follows: $E_{\mathrm{meas}}(\rho^{AB}) = \min_U E_D^{M|AB}(U\rho_1 U^\dagger)$, where the minimization is done over all unitaries which realize some von Neumann measurement on A. Recalling the definition of the one-way information deficit in (2), we present one of our main results.

Theorem 1. If a bipartite state ρ^{AB} has nonzero quantum discord $\delta^{\rightarrow}(\rho^{AB}) > 0$, any von Neumann measurement on A creates distillable entanglement between the measurement apparatus and the total system AB. The minimal distillable entanglement created in a von Neumann measurement on A is equal to the one-way information deficit: $E_{\mathrm{meas}}(\rho^{AB}) = \Delta^{\rightarrow}(\rho^{AB})$.

Proof.—As pointed out in [18], the unitary U must act on states of the form $|0^M\rangle \otimes |i^A\rangle$ as follows: $U(|0^M\rangle \otimes |i^A\rangle) = |i^M\rangle \otimes |i^A\rangle$, where $\{|i^A\rangle\}$ is the measurement basis, and $|i^M\rangle$ are orthogonal states of the measurement apparatus. In general we can always write $\rho^{AB} = \sum_{i,j} |i^A\rangle\langle j^A| \otimes O_{ij}^B$ with O_{ij}^B being operators on the Hilbert space $\mathcal{H}_B$. After the action of the unitary the state becomes $\rho_2 = \sum_{i,j} |i^M\rangle \times \langle j^M| \otimes |i^A\rangle\langle j^A| \otimes O_{ij}^B$. From [19] we know that the distillable entanglement is bounded from below as $E_D^{M|AB}(\rho_2) \geq S(\rho_2^{AB}) - S(\rho_2)$ with $\rho_2^{AB} = \mathrm{Tr}_M[\rho_2]$, and the von Neumann entropy $S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]$. We mention that the same inequality holds for the relative entropy of entanglement defined in [20] as $E_R = \min_{\sigma \in S} S(\rho\|\sigma)$ with the quantum relative entropy $S(\rho\|\sigma) = -\mathrm{Tr}[\rho \log_2 \sigma] + \mathrm{Tr}[\rho \log_2 \rho]$; see [21] for details. Noting that $\rho_2^{AB} = \sum_i \Pi_i^A \rho^{AB} \Pi_i^A$ and $S(\rho_2) = S(\rho_1) = S(\rho^{AB})$ we see $E_D^{M|AB}(\rho_2) \geq S(\sum_i \Pi_i^A \rho^{AB} \Pi_i^A) - S(\rho^{AB})$. On the other hand, we know that E_R is an upper bound on the distillable entanglement [22]. Consider the state $\sigma = \sum_i \Pi_i^M \rho_2 \Pi_i^M$, which is separable with respect to the bipartition $M|AB$. From the definition of the relative entropy of entanglement follows: $E_R^{M|AB}(\rho_2) \leq S(\rho_2\|\sigma)$. It can be seen by inspection that $S(\rho_2\|\sigma) = S(\sum_i \Pi_i^A \rho^{AB} \Pi_i^A) - S(\rho^{AB})$. Thus we proved that $E_D^{M|AB}(\rho_2) = S(\sum_i \Pi_i^A \rho^{AB} \Pi_i^A) - S(\rho^{AB})$ holds for any measurement basis $\{|i^A\rangle\}$. If we minimize this equation over all von Neumann measurements on A, we get the desired result. ■

Note that from the above proof we conclude that $\min_U E_D^{M|AB}(U\rho_1 U^\dagger) = \min_U E_R^{M|AB}(U\rho_1 U^\dagger)$, and thus there does not exist bound entanglement in a partial measurement.

The approach presented so far can also be applied to any other measure of entanglement E, which satisfies the basic axiom to be nonincreasing under local operations and classical communication (LOCC) [20]. In this way we introduce the generalized one-way information deficit as follows:

$$\Delta_E^{\rightarrow}(\rho^{AB}) = \min_U E^{M|AB}(U\rho_1 U^\dagger), \tag{3}$$

where U realizes a von Neumann measurement on A and $\rho_1 = |0^M\rangle\langle 0^M| \otimes \rho^{AB}$. Using Theorem 1 it is easy to see that the generalized one-way information deficit is zero if and only if the state ρ^{AB} has zero quantum discord. This holds if E is zero on separable states only.

In the same way as different measures of entanglement capture different aspects of entanglement, the correspondence (3) can be used to capture different aspects of quantum correlations. Let us demonstrate this by using the geometric measure of entanglement E_G [23] on the right-hand side of (3). As the corresponding measure of quantum correlations, we obtain $\Delta_{E_G}^{\rightarrow}(\rho^{AB}) = \min_{\delta^{\rightarrow}(\sigma^{AB})=0}\{1 - F(\rho^{AB}, \sigma^{AB})\}$ with the fidelity $F(\rho, \sigma) = (\mathrm{Tr}[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}])^2$ [24]. The minimization is done over all states σ^{AB} with zero quantum discord. Thus, this measure captures the geometric aspect of quantum correlations, similarly to the geometric measure of discord presented in [9].

The correspondence (3) also implies that certain properties of entanglement measures are transferred to corresponding properties of quantum correlation measures. This will be demonstrated in the following by finding a class of quantum operations which do not increase $\Delta_E^{\rightarrow}$. This class cannot be equal to the class of LOCC, since $\Delta_E^{\rightarrow}$ can increase under local operations on A. This can be seen by considering the classically correlated state $\rho_{cc} = \frac{1}{2}|0^A\rangle\langle 0^A| \otimes |0^B\rangle\langle 0^B| + \frac{1}{2}|1^A\rangle\langle 1^A| \otimes |1^B\rangle\langle 1^B|$ with $\Delta_E^{\rightarrow}(\rho_{cc}) = 0$. Using only local operations on A it is possible to create states with nonzero deficit $\Delta_E^{\rightarrow}$. Demanding that the subsystem A is unchanged, we are left with quantum operations on B only. In the following we will show that $\Delta_E^{\rightarrow}$ does not increase under arbitrary quantum operations on B, denoted by Λ_B:

$$\Delta_E^{\rightarrow}(\Lambda_B(\rho^{AB})) \leq \Delta_E^{\rightarrow}(\rho^{AB}). \tag{4}$$

Inequality (4) is seen to be true by noting that the entanglement $E^{M|AB}$ does not increase under Λ_B, as it does not increase under LOCC.

We can go one step further by noting that the distillable entanglement is also nonincreasing on average under stochastic LOCC. This captures the idea that two parties cannot share more entanglement on average, if they perform local generalized measurements on their subsystems and communicate the outcomes classically; see [17] for more details. Defining the global Kraus operators describing some LOCC protocol by $\{V_i\}$ with $\sum_i V_i^\dagger V_i = \mathbb{1}$, the probability of the outcome i is given by $q_i = \mathrm{Tr}[V_i \rho V_i^\dagger]$, and the state after the measurement with the outcome i is given by $\sigma_i = V_i \rho V_i^\dagger / q_i$. Then for the distillable entanglement [25] and the relative entropy of entanglement holds [26]

$$\sum_i q_i E(\sigma_i) \leq E(\rho). \tag{5}$$

Inequality (5) implies that the corresponding quantity $\Delta_E^{\rightarrow}$ satisfies the related property

$$\sum_i q_i \Delta_E^{\rightarrow}(\sigma_i^{AB}) \leq \Delta_E^{\rightarrow}(\rho^{AB}), \tag{6}$$

where q_i, σ_i^{AB} are defined as above Eq. (5), and now $\{V_i\}$ are Kraus operators describing a local quantum operation on B. Inequality (6) is seen to be true by using (5) in the definition (3).

In the following we will include the quantum discord $\delta^{\rightarrow}$ into our approach. We call the non-negative quantity

$$P_E(\rho) = E^{M|AB}(\rho) - E^{M|A}(\rho^{MA}) \tag{7}$$

the partial entanglement. It quantifies the part of entanglement which is lost when the subsystem B is ignored; see also Fig. 1. The following theorem establishes a connection between the partial entanglement and the quantum discord.

Theorem 2. The quantum discord of a bipartite state ρ^{AB} is equal to the minimal partial distillable entanglement in a von Neumann measurement on A: $\delta^{\rightarrow}(\rho^{AB}) = \min_U P_{E_D}(U\rho_1 U^\dagger)$. The minimization is done over all unitaries U which realize a von Neumann measurement on A, and $\rho_1 = |0^M\rangle\langle 0^M| \otimes \rho^{AB}$.

Proof.—We note that for any state ρ^{AB} the quantum discord can be written as $\delta^{\rightarrow}(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \min_{\{\Pi_i^A\}}\{S(\sum_i \Pi_i^A \rho^{AB} \Pi_i^A) - S(\sum_i \Pi_i^A \rho^A \Pi_i^A)\}$ with the minimization over all von Neumann measurements on A. To see this we start with the definition of the discord in (1). Then it is sufficient to show that for $p_i = \mathrm{Tr}[\Pi_i^A \rho^{AB} \Pi_i^A]$ and $\rho_i = \Pi_i^A \rho^{AB} \Pi_i^A / p_i$ holds $\sum_i p_i S(\rho_i) = S(\sum_i \Pi_i^A \rho^{AB} \Pi_i^A) - S(\sum_i \Pi_i^A \rho^A \Pi_i^A)$, which can be seen by inspection using the fact that $\{p_i\}$ are eigenvalues of $\sum_i \Pi_i^A \rho^A \Pi_i^A$. Using the same arguments as in the proof of Theorem 1 the desired result follows. ■

Using Theorem 2 we will show that the properties (4) and (6) are also satisfied by the quantum discord. Inequality (4) can be seen to be true by noting that E_D does not increase under LOCC and that Λ_B does not change the state $\mathrm{Tr}_B[U\rho_1 U^\dagger]$. To see that (6) also holds for the quantum discord note that, using the same arguments as in the proof of Theorem 1, we can replace the distillable entanglement E_D by the relative entropy of entanglement E_R in Theorem 2 without changing the statement. Because of convexity of E_R [26], the entanglement $E_R^{M|A}$ is nondecreasing on average under quantum operations on B: $\sum_i q_i E_R^{M|A}(\sigma_i^{MA}) \geq E_R^{M|A}(\rho^{MA})$. This implies that the partial entanglement $P_{E_R}(\rho) = E_R^{M|AB}(\rho) - E_R^{M|A}(\rho^{MA})$ is nonincreasing on average under quantum operations on B. Using this result we see that (6) also holds for the quantum discord.

Theorem 2 allows us to generalize the quantum discord to arbitrary measures of entanglement E in the same way as it was done for the one-way information deficit in (3):

$$\delta_E^{\rightarrow}(\rho^{AB}) = \min_U P_E(U\rho_1 U^\dagger). \tag{8}$$

Using the same arguments as above Eq. (8) we see that the generalized quantum discord $\delta_E^{\rightarrow}$ satisfies the properties (4) and (6) for all measures of entanglement E which are convex and obey (5).

So far we have only considered von Neumann measurements. In the following we will show that our approach is also valid with an alternative definition of the quantum discord [11,12,27]: $\delta_{\mathrm{POVM}}^{\rightarrow}(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \min_{\{M_i^A\}} \sum_i p_i S(\rho_i^B)$, with $\{M_i^A\}$ being a positive operator-valued measure (POVM) on A, $p_i = \mathrm{Tr}[M_i^A \rho^{AB}]$ and $\rho_i^B = \mathrm{Tr}_A[M_i^A \rho^{AB}]/p_i$. The minimization over POVMs can be replaced by a minimization over orthogonal projectors of rank one $\{\Pi_i^{A'}\}$ on an extended Hilbert space $\mathcal{H}_{A'}$ with $\dim\mathcal{H}_{A'} \geq \dim\mathcal{H}_A$ [28]. With this observation we see that all results presented for the quantum discord also hold for the alternative definition of the quantum discord.

In the following we will generalize our approach to multipartite von Neumann measurements on A.

We split the system A into n subsystems: $A = \cup_{i=1}^{n} A_i$. A von Neumann measurement Λ will be called n-partite, if it can be expressed as a sequence of von Neumann measurements Λ_i on each subsystem A_i: $\Lambda(\rho) = \Lambda_1(\ldots \Lambda_n(\rho))$. Now we can introduce the n-partite one-way information deficit $\Delta_n^{\rightarrow}$ and the n-partite quantum discord $\delta_n^{\rightarrow}$ as follows:

$$\Delta_n^{\rightarrow}(\rho^{AB}) = \min_{\Lambda} S(\Lambda(\rho^{AB})) - S(\rho^{AB}), \tag{9}$$

$$\delta_n^{\rightarrow}(\rho^{AB}) = \min_{\Lambda}\{S(\Lambda(\rho^{AB})) - S(\Lambda(\rho^{A}))\} - S(\rho^{AB}) + S(\rho^{A}). \tag{10}$$

Using the same arguments as in the proof of Theorems 1 and 2, we see that $\Delta_n^{\rightarrow}$ quantifies the minimal distillable entanglement between M and AB created in an n-partite von Neumann measurement on A. $\delta_n^{\rightarrow}$ can be interpreted as the corresponding minimal partial distillable entanglement P_{E_D}. We also note that this generalization includes n-partite von Neumann measurements on the total system. This can be achieved by defining A to be the total system. Since $\delta_n^{\rightarrow} = 0$ in this case, the only nontrivial quantity is the generalized information deficit $\Delta_n^{\rightarrow}$. A different approach to extend the quantum discord to multipartite settings was introduced in [29].

In this work we showed that the one-way information deficit is equal to the minimal distillable entanglement between the measurement apparatus M and the system AB which has to be created in a von Neumann measurement on A. The quantum discord is equal to the corresponding minimal partial distillable entanglement. Our approach can also be applied to any other measure of entanglement, thus defining a class of quantum correlation measures. This correspondence allows us to translate certain properties of entanglement measures to corresponding properties of quantum correlation measures. It may lead to a better understanding of the quantum discord and related measures of quantum correlations, since it allows us to use the great variety of powerful tools developed for quantum entanglement. We found a class of quantum operations which do not increase the generalized versions of the one-way information deficit and the quantum discord. We also generalized our results to multipartite settings.

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Note added.—Recently an alternative approach to connect the entanglement to quantum correlation measures was presented in [30]. There the authors show that nonclassical correlations in a multipartite state can be used to create entanglement in an activation protocol.

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