Optical Phase Estimation in the Presence of Phase Diffusion

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The measurement problem for the optical phase has been traditionally attacked for noiseless schemes or in the presence of amplitude or detection noise. Here we address the estimation of phase in the presence of phase diffusion and evaluate the ultimate quantum limits to precision for phase-shifted Gaussian states. We look for the optimal detection scheme and derive approximate scaling laws for the quantum Fisher information and the optimal squeezing fraction in terms of the total energy and the amount of noise. We also find that homodyne detection is a nearly optimal detection scheme in the limit of very small and large noise.

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The estimation of the optical phase in quantum mechanics is a long-standing problem with both fundamental and technological implications. The attempts to define a Hermitian phase operator are the subject of extensive literature [1] and several experimental protocols to estimate the value of the optical phase have been also proposed and demonstrated, in particular, using different quantum strategies and interferometric setups [2–7], which have been shown to beat the standard quantum limit [8–11]. More recently, the ultimate bounds to the precision of the phase estimation with Gaussian states have been discussed [12,13] using local quantum estimation theory. A squeezed vacuum state has been shown to be the most sensitive at fixed energy and two measurement schemes have been proposed to attain the Heisenberg limit.

The estimation of the optical phase is also relevant for optical communication schemes where information is encoded in the phase of traveling pulses. In such a context the receiver has to decode information that is unavoidably degraded by different sources of noise, which have to be taken into account in the quantum estimation problem. So far, only amplitude and/or detection noise have been taken into account in the analysis of quantum phase estimation, e.g., imperfect photodetection in the measurement stage, or amplitude noise in interferometric setups [14–18]. The role of phase-diffusive noise in phase estimation has been investigated for qubit [19,20] and in part for condensate systems [21,22], while no similar analysis has been done for a continuous-variable system. Phase-diffusive noise is the most detrimental for phase estimation since it destroys the off-diagonal elements of the density matrix. Moreover, any quantum state that is unaffected by phase diffusion, is also invariant under a phase shift, and thus is totally useless for phase estimation.

In this Letter we address phase estimation in the presence of phase diffusion, seek for the optimal Gaussian states, and evaluate the ultimate quantum limits to precision of phase estimation. We also investigate whether the ultimate performances may be achieved with a feasible detection scheme and found that homodyne detection is nearly optimal for very small and large amounts of noise.

When a physical parameter is not directly accessible one has to resort to indirect measurements. Let us denote by ϕ the quantity of interest, X the measured observable, and $\chi = (x_1, \ldots, x_M)$ the observed sample. The estimation problem amounts to finding an estimator, that is a map $\hat{\phi} = \hat{\phi}(\chi)$ from the set of the outcomes to the space of parameters. Classically, optimal estimators are those saturating the Cramér-Rao inequality $\mathrm{Var}(\phi) \geq [MF(\phi)]^{-1}$ which bounds from below the variance $\mathrm{Var}(\phi) = E[\hat{\phi}^2] - E[\hat{\phi}]^2$ of any unbiased estimator of the parameter ϕ. M is the number of measurements and $F(\phi)$ is the Fisher information (FI) $F(\phi) = \int dx p(x|\phi)[\partial_\phi \ln p(x|\phi)]^2$, where $p(x|\phi)$ is the conditional probability of obtaining the value x when the parameter has the value ϕ. The quantum Cramér-Rao bound [23–26] is obtained starting from the Born rule $p(x|\phi) = \mathrm{Tr}[\Pi_x \varrho_\phi]$ where $\{\Pi_x\}$ is the probability operator-valued measure (POVM) describing the measurement and ϱ_ϕ the density operator, labeled by the parameter of interest. Upon introducing the symmetric logarithmic derivative (SLD) L_ϕ as the operator satisfying $2\partial_\phi \varrho_\phi = L_\phi \varrho_\phi + \varrho_\phi L_\phi$ one proves that the FI is upper bounded by the quantum Fisher information (QFI) [24] $F(\phi) \leq H(\phi) \equiv \mathrm{Tr}[\varrho_\phi L_\phi^2]$. In turn, the ultimate limit to precision is given by the quantum Cramér-Rao bound $\mathrm{Var}(\phi) \geq [MH(\phi)]^{-1}$. The family of states we are going to deal with is a unitary one $\varrho_\phi = U_\phi \varrho_0 U_\phi^\dagger = \sum_k \lambda_k |\lambda_k(\phi)\rangle\langle\lambda_k(\phi)|$, where $|\lambda_k(\phi)\rangle = U_\phi|\lambda_k\rangle$ and $U_\phi = \exp\{-i\phi G\}$ describes a phase shift with the single-mode number operator $G = a^\dagger a$ as the generator. In this case the SLD may be written as $L_\phi = U_\phi L_0 U_\phi^\dagger$, where L_0 is independent on ϕ. The corresponding QFI does not depend on the parameter ϕ, and reads

$$H = \mathrm{Tr}[\varrho_0 L_0^2] = 2\sum_{n\neq m} \frac{(\lambda_n - \lambda_m)^2}{\lambda_n + \lambda_m} |\langle\lambda_n|G|\lambda_m\rangle|^2. \quad (1)$$

Phase diffusion for a continuous-variable system is described by the master equation $\dot{\varrho} = \Gamma\mathcal{L}[a^\dagger a]\varrho$, where $\mathcal{L}[O]\varrho = 2O\varrho O^\dagger - O^\dagger O\varrho - \varrho O^\dagger O$. The solution for an initial state $\varrho(0)$ is given by $\varrho(t) = \mathcal{N}_\Delta(\varrho(0)) = \sum_{n,m} e^{-\Delta^2(n-m)^2}\varrho_{n,m}(0)|n\rangle\langle m|$ where $\Delta \equiv \Gamma t$, Γ is the noise amplitude and $\varrho_{n,m}(0) = \langle n|\varrho(0)|m\rangle$. The diagonal elements ϱ are left unchanged and energy is conserved, whereas the off-diagonal ones are progressively destroyed.

We assume that phase noise occurs between the application of the unknown phase shift U_ϕ and the detection of the signal, and address quantum estimation of a phase shift applied to pure single-mode Gaussian states $|\psi_G\rangle = D(\alpha)S(r)|0\rangle$ where $S(r) = \exp\{(r/2)(a^2 - a^{\dagger 2})\}$ is the squeezing operator, $D(\alpha) = \exp\{\alpha(a^\dagger - a)\}$ the displacement operator, being $r, \alpha \in \mathbb{R}$. Our aim is to determine the ultimate bound to precision for a generic pure Gaussian probe and then look for the optimal one by maximizing the QFI over the state parameters.

The mixed non-Gaussian state that is being measured is given by

$$\varrho_\phi(t) = \mathcal{N}_\Delta(U_\phi|\psi_G\rangle\langle\psi_G|U_\phi^\dagger) = U_\phi\mathcal{N}_\Delta(|\psi_G\rangle\langle\psi_G|)U_\phi^\dagger,$$

where the second equality holds since the superoperator $\mathcal{L}[a^\dagger a]$ and the phase-shift operator U_ϕ commute. Our estimation problem thus corresponds to the case of a unitary family described above, with the input mixed state given by $\mathcal{N}_\Delta(|\psi_G\rangle\langle\psi_G|)$. In order to evaluate the corresponding QFI one writes ϱ_ϕ in its diagonal form $\varrho_\phi = \sum_n \lambda_n|\lambda_n(\phi)\rangle\langle\lambda_n(\phi)| = \sum_n \lambda_n U_\phi|\lambda_n\rangle\langle\lambda_n|U_\phi^\dagger$, where $|\lambda_n(\phi)\rangle$ and $|\lambda_n\rangle$ are, respectively, the eigenvectors of ϱ_ϕ and of $\mathcal{N}_\Delta(|\psi_G\rangle\langle\psi_G|)$ corresponding to the eigenvalues λ_n, which are in fact left unchanged by the phase-shift operation. By decomposing $|\lambda_n\rangle = \sum_k r_{nk}|k\rangle$ in the Fock basis and by substituting this into the eigenvalues equation $\mathcal{N}_\Delta(|\psi_G\rangle\langle\psi_G|)|\lambda_n\rangle = \lambda_n|\lambda_n\rangle$ we have $\langle n|\psi_G\rangle\langle\psi_G|l\rangle e^{-\Delta^2(n-k)^2} r_{qk} = \lambda_q r_{qn} \forall n$. Moreover, since $a^\dagger a|\lambda_n\rangle = \sum_k k r_{nk} e^{ik\phi}|k\rangle$, we have that $|\langle\lambda_m|a^\dagger a|\lambda_n\rangle|^2 = |\sum_k k r_{mk} r_{nk}|^2$. After evaluating the QFI using the above formulas one sees that it depends only on the eigenvalues λ_n and on the components of the eigenvectors r_{nk} which, being ϕ a unitary parameter, do not depend on the parameter itself. The explicit values of λ_n and r_{nk} have been obtained by performing numerical diagonalization.

The vanishing of the off-diagonal matrix elements is governed by the product between Δ^2 and $(n-m)^2$, i.e., the squared difference between the Fock indices. Besides, for a pure Gaussian state, the presence of nonzero (non-negligible) off-diagonal elements is somehow ruled by the average photon number $N = \langle a^\dagger a\rangle$ and thus we roughly expect the QFI to somehow depend on the quantity $\xi = N\Delta$. Pure Gaussian states may be conveniently parametrized by the average photon number N and the corresponding squeezing fraction β, in formula $N = \sinh^2 r + |\alpha|^2$ and $\beta = \sinh^2 r/N$, and thus the QFI will be a function of the three parameters N, β, and Δ.

We start our analysis by evaluating the QFI at fixed noise Δ. We consider four values of the maximum energy $N_{\max} = \langle a^\dagger a\rangle_{\max} = \{10, 15, 20, 30\}$ (with 10 steps on intermediate energies N) and different values of the noise parameter Δ. The values of Δ are chosen such that we can find points corresponding to fixed values of ξ. The curves are built by looking for the optimal pure Gaussian state, i.e., maximizing the QFI as a function of the squeezing fraction β, for any fixed value of the energy N and of the noise parameter Δ.

The values of the optimal squeezing fraction $\beta_{\rm opt} = \beta_{\rm opt}(N, \Delta)$ and of the corresponding QFI $H(N, \beta_{\rm opt}, \Delta)$ have been numerically evaluated and are reported in Figs. 1 and 2, respectively. As we can see in Fig. 1, for a low level noise the squeezing fraction is approaching one, and thus the optimal probe state is the squeezed vacuum state, as it happens in the noiseless case [12]. As far as the noise Δ increases the squeezing fraction decreases as a function of the average number of photons. This means that for increasing values of the noise and of the energy, it is more convenient to employ the energy in increasing the coherent amplitude rather than squeezing of the probe. Let us now focus on the behavior of the QFI $H(N, \beta_{\rm opt}, \Delta)$. In the left panel of Fig. 2, we report the typical behavior of the QFI as a function of N and for different values of Δ. The QFI increases by increasing the average photon number N, and decreases with the noise parameter Δ. For the lowest value of Δ, we also observe that the noiseless limit $H(N, \beta = 1, \Delta = 0) = 8(N^2 + N)$ [12] is approached, at least for N not too large.

As we have already mentioned above we expect the product $\xi = N\Delta$ to play a role in the estimation properties. In fact, by exploring a large range of values for N and Δ a scaling law emerges from numerical analysis, which may be written as

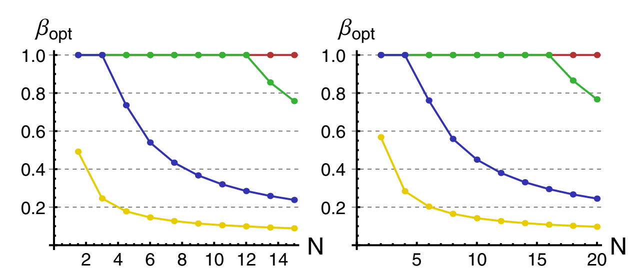

FIG. 1 (color online). Optimal squeezing fraction β as a function of the average photon number N and for different values of Δ^2. (Left): from top to bottom $\Delta^2 = \{2.0\times10^{-5}, 2.0\times10^{-4}, 2.0\times10^{-3}, 2.0\times10^{-2}\}$. (Right): from top to bottom $\Delta^2 = \{1.125\times10^{-5}, 1.125\times10^{-4}, 1.125\times10^{-3}, 1.125\times10^{-2}\}$.

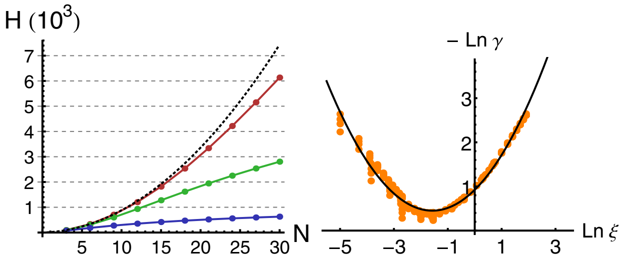

FIG. 2 (color online). Left panel: QFI of optimized pure input Gaussian states as a function of the average photon number N and for different values of the noise parameter Δ; from top to bottom $\Delta^2 = \{5.0 \times 10^{-6}, 5.0 \times 10^{-5}, 5.0 \times 10^{-4}\}$. The black dotted line is the QFI for the noiseless case $H(N, \beta = 1, \Delta = 0) = 8(N^2 + N)$. Right panel: (points) $-\ln\gamma(\xi)$ as a function of $\ln\xi$, $\xi \equiv N\Delta$, with $N \leq 30$ and $10^{-3} \leq \Delta \leq 1$. The black curve is a best fit with functional form $\gamma(\xi) \propto \xi^{-b}\exp(-a\ln^2\xi)$.

$$H(N, \Delta) \simeq k^2 H(N/k, k\Delta). \tag{2}$$

That is, $H(N, \Delta) = N/\Delta\gamma(\xi) = N^2\gamma(\xi)/\xi = \xi\gamma(\xi)/\Delta^2$ where $0 < \gamma(\xi) < 1$ is a universal function independent on Δ and N. The larger is ξ the more accurate is the scaling law. The scaling is illustrated in the right panel of Fig. 2 where we report the quantity $-\ln\gamma(\xi)$ as a function of $\ln\xi$ (points) together with a two-parameter fit (black curve) of the form $\gamma(\xi) \propto \xi^{-b}\exp(-a\ln^2\xi)$, that provides a good representation of data. Using the above results, the quantum Cramèr-Rao bound for the precision of an optimal estimator of the phase shift may be written as $\mathrm{Var}(\phi) \gtrsim \frac{\Delta}{\gamma(\xi)N} = \frac{\xi}{\gamma(\xi)N^2}$. For small values of ξ the quantity $\xi\gamma(\xi)$ is of order of unity and thus Heisenberg limit $\mathrm{Var}(\phi) \sim N^{-2}$ in precision may be achieved [27]. We also found that another scaling law, though less accurate, holds for the optimal squeezing fraction

$$\beta_{\mathrm{opt}}(N, \Delta) \simeq \beta_{\mathrm{opt}}(N/k, k\Delta). \tag{3}$$

Though based on a physical and mathematical justification, we cannot expect these scaling laws to be exact due to the non-Gaussianity of the state. However, they give a useful and practical receipt to compare and predict phase-estimation performances in different regimes of energy and noise. In the left panel of Fig. 3 we show the behavior of the quantum Fisher information at fixed average photon number as a function of Δ. We notice that the $H(N, \Delta)$ decreases exponentially with the phase noise and that higher values of N correspond to higher values of H. We have also evaluated the behavior of the QFI for some two-mode entangled signals as entangled coherent and NOON states and found that the QFI, which is initially smaller than for optimized Gaussian states, remains smaller for any value of the noise parameter; i.e., these classes of two-mode signals do not offer large robustness.

In the noiseless case ($\Delta = 0$) homodyne detection on squeezed vacuum states is optimal [12]; its Fisher information F is equal to the QFI, $H(N) = 8(N^2 + N)$.

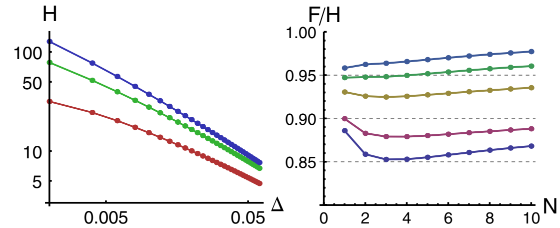

FIG. 3 (color online). (Left): Log-log plot of the QFI for optimized pure input Gaussian states as a function of the noise parameter Δ for different values of the average photon number. From bottom to top: $N = \{2, 5, 10\}$. (Right): Ratio between the FI of homodyne detection on coherent states and the corresponding QFI, as a function of the number of photons of the probe states and for different values of Δ. From bottom to top: $\Delta^2 = \{0.5, 1.0, 1.5, 2.0, 5\}$.

The question thus arises of whether or not this result also holds in the presence of phase diffusion. Our numerical findings show that this is true for a very small amount of noise, i.e., $\Delta \ll 1$, whereas for increasing Δ the ratio F/H moves away from unity quite quickly. On the other hand, one can see that for high values of Δ, basically when coherent states are the optimal probe states maximizing the QFI, homodyne detection of the quadrature $X = (a + a^\dagger)/2$ is again nearly optimal; i.e., its Fisher information again approaches the value of the QFI evaluated in same conditions. In the right panel of Fig. 3 we plot the ratio between the Fisher information of homodyne detection and the corresponding QFI: by increasing the noise Δ the ratio increases towards optimality ($F/H = 1$). This may be understood looking at the behavior of quadrature fluctuations $\Delta X_\theta^2 = \langle X_\theta^2 \rangle - \langle X_\theta \rangle^2$ since the smaller is ΔX_θ^2 for a certain quadrature X_θ, the more precise is the estimation of the phase shift through this quadrature. In Fig. 4, we report a contour plot of $\log\Delta X_\theta^2$ as a function of the squeezing fraction of the input state β and the quadrature phase θ for different values of Δ and of the overall energy N. We see that for low noise, i.e., $\Delta \ll 1$, minimum fluctuations are obtained for the quadrature $\theta = \pi/2$ and for a squeezed vacuum state ($\beta = 1$), whereas after a certain energy-dependent threshold level of noise $\Delta^* \equiv \Delta^*(N)$, we have a jump and the minimum fluctuations are achieved by measuring the X quadrature ($\theta = 0$) on coherent probes ($\beta = 0$). This behavior is different compared to the behavior we have obtained for the QFI; see Fig. 1. There, for intermediate values of Δ, the optimal squeezing fraction decreases monotonically from $\beta = 1$ to $\beta = 0$, whereas here we have only the extreme values. This exactly corresponds to the result discussed above: homodyne detection, as far as we tune accordingly the measured quadrature, is optimal for very low noise with squeezed vacuum probes ($\beta = 0$), and for large noise with coherent probes ($\beta = 1$), while for intermediate values of Δ homodyne detection is far from optimality. Overall, we have that homodyne detection provides nearly optimal phase estimation for either very small or large phase

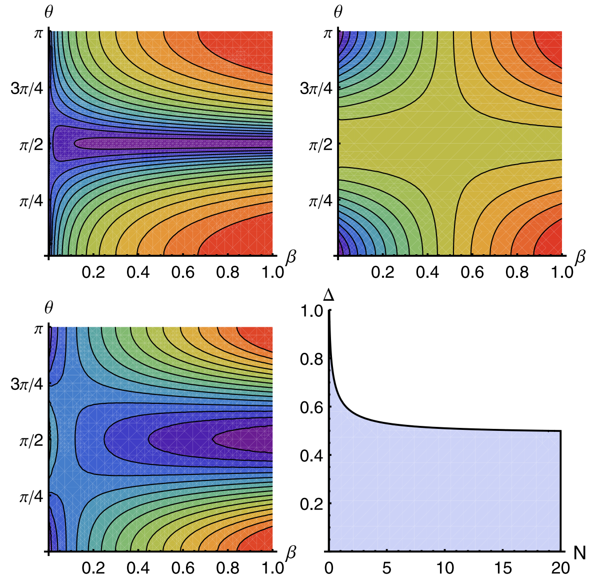

FIG. 4 (color online). Quadrature fluctuations ΔX_θ^2 as a function of the squeezing fraction of the input state β and of the phase θ for different values of the noise amplitude Δ and the overall energy N. Top left: $N = 10$ and $\Delta = 0.1$; top right: $N = 10$ and $\Delta = 0.6$; bottom left: $N = 0.1$ and $\Delta = 0.1$. Darker regions correspond to smaller ΔX_θ^2. The plot in the bottom right panel illustrates the threshold $\Delta^*(N)$ between the two regions where minimum fluctuations are achieved for $\beta = 1$, $\theta = \pi/2$ (gray area) and $\beta = 0$, $\theta = 0$, respectively.

diffusion, whereas it is still an open problem to find a feasible measurement attaining the ultimate precision for a generic value of the phase-diffusion noise parameter Δ.

In conclusion, we have attacked the problem of finding the optimal way to estimate a phase shift in the presence of phase diffusion and we have obtained the ultimate quantum limits to precision for phase-shifted Gaussian states. By an extensive numerical analysis we have obtained approximate scaling laws for both the quantum Fisher information and the optimal squeezing fraction in terms of the overall total energy and the amount of noise. We also found that homodyne detection is a nearly optimal detection scheme for very small or large noise. Our results go beyond the traditional analysis of the quantum phase measurement problem and may be relevant for the development of phase-shift keyed optical communication schemes [28].

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