Pauli Blocking Effect on Efimov States near a Feshbach Resonance

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In this Letter we study the effect of Pauli blocking on Efimov states in a quantum Fermi gas and illustrate that the universal Efimov potential is altered at large distances. We obtain the universal spectrum flow of Efimov trimers when the Fermi density is varied and further consider the effect of scattering of trimers by the Fermi sea. We argue that the universal flow is robust against fluctuating particle-hole pairs that result in an infrared catastrophe in impurity problems.

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Recently, universal three-body Efimov structures that were first proposed in the 1970s have been successfully probed and determined in a series of remarkable cold-atom experiments on inelastic loss spectra [1–13]. The current data that cover an impressive range of scattering lengths quite conclusively demonstrate the universality of the Efimov structures [12,13]. The spectacular progress made on this subject has further illustrated that the physics of cold atoms at submicrokelvin temperatures can shed light on fundamental issues in other systems at very different energy scales, in this case the few-body structures of nuclei.

Although Efimov physics [1,2] and the theory of loss rates [3,4] have so far been quite successful in explaining many aspects of the data, in experiments on cold gases there are always many-body backgrounds. Logically, it is important to understand how few-body states respond to the presence of a quantum many-body background below degeneracy temperatures. For instance, can Efimov trimers survive a many-body background and how are they affected when scattered by the background? An equally important and challenging question is what kind of many-body correlations can be induced by the universal few-body states in a quantum gas or mixture. In this Letter, we attempt to answer one of these questions, and particularly to examine the effect of Pauli blocking of scattering or open-channel Fermi atoms on the three-body Efimov states. To answer this question, we solve the three-body spectrum in the presence of a Fermi sea and focus on the effect of Pauli blocking on the Efimov physics. These spectra are known to be one of the cornerstones for the theory of three-body recombination and the loss spectrum of metastable quantum gases. Our results can also be applied to study the dynamics during the initial stage of recombination and estimate the lifetime of quantum gases.

In the following, we consider that two identical heavy Bose atoms with mass M have an interspecies resonance with a light Fermi atom with mass $m (\ll M)$ in the presence of a Fermi sea of light atoms. In the Born-Oppenheimer approximation, we can analyze the fast motion of the light atom assuming that the two slow heavy atoms are a distance R apart. In the absence of a Fermi sea, simultaneous near-resonance scattering of the light atom by the two heavy atoms induces a bound state, and the binding energy at resonance is equal to $\hbar^2\Omega^2/2\mu R^2$, where $\Omega = 0.567$ is the root of $\Omega = e^{-\Omega}$ (see below) and $\mu = mM/(m+M)$ is the reduced mass of a light-heavy subsystem. The bound state therefore glues together the two heavy atoms and yields an attractive long range potential $-\hbar^2\Omega^2/2\mu R^2$. This universal $1/R^2$ potential plays a paramount role in Efimov physics and results in the spectacular universal hierarchy structures in three-body spectra [1,2]. We then investigate the effective attractive potential between two heavy atoms in the presence of a Fermi sea with Fermi momentum $\hbar k_F$. The interspecies interactions can be treated as zero-range ones [14,15], so that for a light atom interacting with two heavy atoms at $\pm\mathbf{R}/2$, $V(\mathbf{r}) = V_0[\delta(\mathbf{r}-\mathbf{R}/2)+\delta(\mathbf{r}+\mathbf{R}/2)]$ where

$$\frac{1}{V_0} = \frac{\mu}{2\pi a_{HL}\hbar^2} - \frac{1}{(2\pi)^3}\int d\mathbf{k}\,\frac{1}{\epsilon_k^\mu}. \tag{1}$$

Equation (1) relates the contact interaction strength V_0 to the scattering length a_{HL} via a standard regularization procedure [16]. Here $\epsilon_k^\mu = \hbar^2k^2/2\mu$ and now the reduced mass $\mu \sim m$ since M is much heavier than m. The Schrödinger equation for $\phi(\mathbf{k})$, the momentum-space wave function for the light atom, is

$$(\epsilon_k^m - E_L)\phi(\mathbf{k}) = \frac{-2V_0}{(2\pi)^3}\int d\mathbf{k}'\cos\frac{(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}}{2}\,\phi(\mathbf{k}'), \tag{2}$$

where $\epsilon_k^m = \frac{\hbar^2k^2}{2m}$ is the kinetic energy for the light atom, and $\phi(\mathbf{k}) = 0$ when $\mathbf{k}$ is within a spherical Fermi surface of radius k_F due to the Pauli blocking effect. We find that the binding energy E_L is given by

$$\frac{2}{\pi}\sqrt{-u_L}\arctan\left(\frac{\sqrt{-u_L}}{k_F}\right) + \frac{2k_F}{\pi} = \frac{1}{a_{HL}} + \frac{2}{\pi R}\int_{k_F}^{\infty} dq\,\frac{q\sin qR}{q^2-u_L}, \tag{3}$$

where we have set $u_L = 2mE_L/\hbar^2$. This equation is valid for $-\infty < u_L < k_F^2$, and has a unique solution for arbitrary

$a_{\rm HL}$. One can easily verify that the solution can be written as $u_{\rm L}(a_{\rm HL}, R, k_{\rm F}) = -k_{\rm F}^2 g(k_{\rm F}a_{\rm HL}, k_{\rm F}R)$, where g is a dimensionless scaling function.

In the absence of a Fermi sea or when $k_{\rm F} \to 0$, $u_{\rm L}$ satisfies the following simple equation: $\sqrt{-u_{\rm L}} = \frac{1}{a_{\rm HL}} + \frac{e^{-\sqrt{-u_{\rm L}}R}}{R}$. So for a positive scattering length and $R = \infty$, $E_{\rm L} = -\hbar^2/2ma_{\rm HL}^2$ and the light atom forms a two-body bound state with one of the heavy atoms. However, when $R \ll |a_{\rm HL}|$ the bound state is severely affected by the second heavy atom and $E_{\rm L} = -\hbar^2\Omega^2/2mR^2$, even when $a_{\rm HL}$ is negative. One can also show that right at resonance, as a result of nonlinear interference between waves coming off the two heavy atoms, the effective scattering length for the light atom is proportional to R, the distance between the two heavy atoms.

Now we turn to the effect of a finite density of fermions and focus on the resonant case where $1/a_{\rm HL} = 0$; in this limit g is a function of $k_{\rm F}R$ only. At short distances when $k_{\rm F}R \ll 1$, the effective attractive potential $u_{\rm L} = -\Omega^2/R^2$ and is determined by the motion of a single light atom. However, this universal behavior is completely changed when $k_{\rm F}R$ becomes order of unity or larger and the collective effect becomes important. First at $k_{\rm F}R = 0.799$, the bound state energy $E_{\rm L}$ increases to zero. After this point, $E_{\rm L}$ continues to rise until it eventually reaches a maximum and then decreases, settling into a pattern of oscillations around a saturation value; its asymptotic behavior is

$$u_{\rm L}(k_{\rm F}R \to \infty) \to u_{\rm L}^{\infty}\left(1 - \frac{\cos k_{\rm F}R}{(k_{\rm F}R)^2}\right), \tag{4}$$

where $u_{\rm L}^{\infty} \approx 0.695k_{\rm F}^2$. See Fig. 1 where details are shown numerically. Note that $\hbar^2 u_{\rm L}^{\infty}/2\mu$ is identical to the two-body binding energy at resonance and therefore represents the atom-dimer threshold for trimers.

To understand the Pauli blocking effect on trimers, we adopt a $\mathbf{k}$-space approach instead of employing the usual

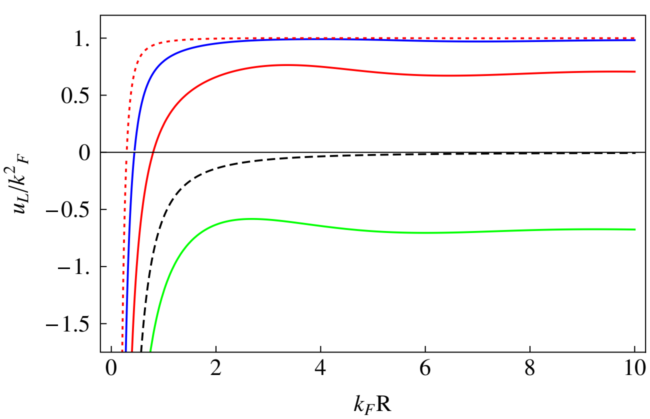

FIG. 1 (color online). Effective potentials $u_{\rm L}/k_{\rm F}^2$ versus $k_{\rm F}R$. From top to bottom we plot the potential for $k_{\rm F}a_{\rm HL} = -1, \infty, 1$. At resonance, $u_{\rm L}(R) = -\Omega^2/R^2$ if the Pauli blocking effect is absent (dashed line); the dotted line is the estimated potential due to the scattering of trimers by the Fermi sea (see discussion near the end of this Letter).

hyperspherical coordinates [1]. The Schrödinger equation for a three-body wave function $\Phi(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is

$$\begin{aligned}(\epsilon^m_{\mathbf{k}_1} + \epsilon^M_{\mathbf{k}_2} + \epsilon^M_{\mathbf{k}_3} - E)\Phi(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= \frac{-V_0}{(2\pi)^3}\int d\mathbf{q}[\Phi(\mathbf{k}_1 - \mathbf{q}, \mathbf{k}_2 + \mathbf{q}, \mathbf{k}_3) \\ &\quad + \Phi(\mathbf{k}_1 - \mathbf{q}, \mathbf{k}_2, \mathbf{k}_3 + \mathbf{q})],\end{aligned} \tag{5}$$

where $\mathbf{k}_1$ is the momentum of the light atom, $\mathbf{k}_2$ and $\mathbf{k}_3$ are the momenta for the heavy atoms, and $\epsilon^m_{\mathbf{k}_1} = \hbar^2\mathbf{k}_1^2/2$ m, $\epsilon^M_{\mathbf{k}_2} = \hbar^2\mathbf{k}_2^2/2$ M. $\Phi(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ vanishes when $\mathbf{k}_1$ is within the spherical Fermi surface of radius $k_{\rm F}$. Furthermore, $\Phi(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \Phi(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2)$ because of the Bose atom exchange statistics. For now we will consider the zero-total-momentum subspace and further introduce a wave function for one heavy atom relative to the heavy-light dimer formed by the other two atoms:

$$\Psi_{\rm AD}(\mathbf{k}_2) = \int_{|\mathbf{q}|>k_{\rm F}} d\mathbf{q}\Phi(\mathbf{q}, \mathbf{k}_2 - \mathbf{q}, -\mathbf{k}_2).$$

The resulting function $\Psi_{\rm AD}(\mathbf{k})$ is shown to satisfy

$$\begin{aligned}\left(\frac{(2\pi)^3}{V_0} + \int' d\mathbf{q}\frac{1}{\epsilon^\mu_{\mathbf{q}} + \epsilon^\gamma_{\mathbf{k}} - E}\right)\Psi_{\rm AD}(\mathbf{k}) \\ = -\int' d\mathbf{q}\frac{\Psi_{\rm AD}(-\mathbf{q} - \frac{\mu}{m}\mathbf{k})}{\epsilon^\mu_{\mathbf{q}} + \epsilon^\gamma_{\mathbf{k}} - E},\end{aligned} \tag{6}$$

where $\gamma = M(M + m)/(2M + m)$ is the reduced mass for collisions between a heavy atom and a heavy-light dimer; also $\epsilon^\gamma_{\mathbf{k}} = \hbar^2\mathbf{k}^2/2\gamma$. Integrals $\int'$ are over a region defined as $|-\frac{\mu}{M}\mathbf{k} + \mathbf{q}| > k_{\rm F}$ to exclude the occupied states. When $\frac{\mu}{m}$ approaches 1, or in the Born-Oppenheimer approximation, we can integrate out the fast degrees of the light atom and map the three-body problem to a simple equation for the dimer and heavy atom:

$$\begin{aligned}\left(\frac{2}{\pi}\sqrt{\frac{k^2}{\beta^2} - u_{\rm B}}\arctan\left(\frac{\sqrt{\frac{k^2}{\beta^2} - u_{\rm B}}}{k_{\rm F}}\right) + \frac{2k_{\rm F}}{\pi} - \frac{1}{a_{\rm HL}}\right)\Psi_{\rm AD}(\mathbf{k}) \\ = \frac{1}{2\pi^2}\int_{q>k_{\rm F}} d\mathbf{q}\frac{\Psi_{\rm AD}(\mathbf{q} - \mathbf{k})}{q^2 + k^2/\beta^2 - u_{\rm B}}.\end{aligned} \tag{7}$$

Equation (7) is valid for $-\infty < u_{\rm B} < k_{\rm F}^2$. Here $u_{\rm B} = 2\mu E/\hbar^2$ and $\beta = \sqrt{\gamma/\mu}$ which is much larger than 1 in our case. The kinetic term (the square-root term on the left-hand side of the equation) scales linearly as a function of k for $k \gg \beta\sqrt{u_{\rm B}}$, reflecting the composite nature of the dimer at short distance; this peculiar structure was appreciated in earlier studies on dimer-atom scattering [14,17]. One can also write down the corresponding differential equation for the Fourier transformed wave function $\Psi_{\rm AD}(\mathbf{r}) = \int d\mathbf{k}\Psi_{\rm AD}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})$. In the limit where β is infinite, Eq. (7) for $\Psi_{\rm AD}(\mathbf{r} = \mathbf{R})$ is equivalent to Eq. (3) for two heavy atoms at a fixed distance $|\mathbf{R}| = R$ apart.

To obtain the spectrum flow of Efimov states, we carry out a WKB analysis of the wave function

$\Psi_{AD}(\mathbf{r}) = \exp[i\beta S_0(r) + iS_1(r)]$ and the bound state energies u_B; the WKB approach is valid as far as β ($\sim \sqrt{M/2m}$) is much larger than unity. By comparing the resulting equation for S_0, derived from Eq. (7), to Eq. (3), one further establishes that S_0 is a simple function of $u_L(r)$. For the S-wave channel, $\Psi_{AD}(r) = \frac{e^{iS_1(r)}}{r} \sin[\beta \int_r^{r_1} dr' \sqrt{u_B - u_L(r)} + \delta]$ for $r < r_1$, where r_1 is the first semiclassical turning point defined by $u_L(r_1) = u_B$ [18] and δ is a phase shift calculated below. When r approaches zero, the phase integral has a logarithmic divergence for any value of k_F, as seen in the hyperspherical approach to the three-atom problem. Without losing generality, we proceed by introducing a Bethe-Peierls boundary condition at $r = r_0$ to take into account a hard-core repulsion with range r_0 that is much shorter than the Fermi wavelength $2\pi/k_F$. The quantization condition is

$$w\left(k_F r_0, \frac{u_n}{k_F^2}\right) = \int_{r_0}^{r_1} dr' \sqrt{u_n - u_L(r')} = \frac{1}{\beta}(n\pi - \delta), \quad (8)$$

where $n = 1, 2, 3, \ldots$ are the indices of the eigenvalues. Since $u_L(r)/k_F^2$ is only a function of $k_F r$, the dimensionless phase w is only a function of $k_F r_0$ and u_B/k_F^2. δ can be obtained by applying the standard matching formulas for the Schrödinger equation, and if there is a single turning point we get $\delta = \pi/4$ [19].

The simplest case is where $k_F = 0$. Then there are infinitely many solutions that accumulate near zero energy, and $u_n/u_{n+1} \to e^{(2\pi)/(\Omega\beta)}$ as $n \to \infty$, with $e^{\pi/\Omega} \approx 254.5$ [1,2]. For a nonzero fermion density, we introduce three dimensionless parameters: $X = k_F r_0$, $Y = u_B/k_F^2$, and $Z = \mathrm{sign}(Y)\sqrt{|Y|}X$. Our results on the spectrum flow are presented in Fig. 2 in the Z-X plane, where each eigenvalue trajectory shows how the solution to Eq. (8) for fixed n varies as we increase X. For fixed nonzero X, the ratio u_n/u_{n+1} depends on n and approaches the universal value of $\exp(2\pi/\Omega\beta)$ only for low lying states with n much less than $N(X)$, where $N(X)$ denotes the number of Efimov states at a given density X. Since $N(X)$ is a rapidly decreasing function of X, the universal value can only be attained when X is small. As X increases, each eigenvalue increases towards u_L^∞, the dimer-atom threshold, eventually reaching this value and disappearing from the spectrum; the eigenvalue trajectories terminate along the ray $Z = X\sqrt{u_L^\infty/k_F^2}$ in the Z-X plane. The termination point X_n^∞ at which the nth eigenvalue reaches the dimer-atom threshold is given by the solution to

$$w(X_n^\infty, Y^\infty) = \frac{1}{\beta}(n\pi - \delta^\infty) \quad (9)$$

where $Y^\infty = u_L^\infty/k_F^2 = 0.695$ and δ^∞ is the phase shift at $Y = Y^\infty$.

As far as X is much less than unity, the termination points have certain universal properties. Indeed, as X approaches 0, $w(X, Y)$ approaches an asymptotic form $w(X, Y) \to -\Omega \log X + v(Y)$ where $v(Y)$ is a regular function of Y

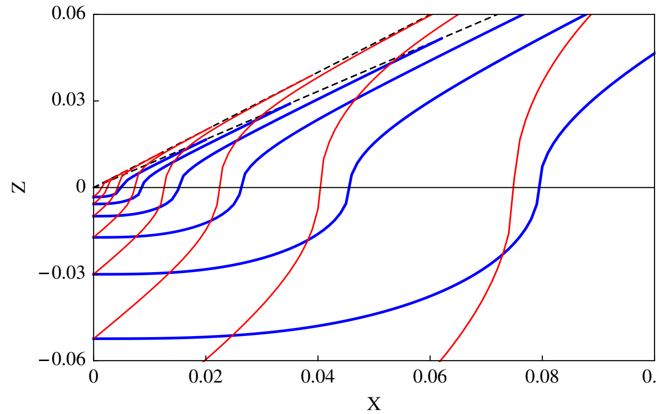

FIG. 2 (color online). A subset of eigenvalue trajectories at resonance ($a_{HL} = \infty$) in the $Z = \mathrm{sgn}(u_B)\sqrt{|u_B|}r_0$ versus $X = k_F r_0$ plane; the thin red trajectories are for trimers dressed in fluctuating particle-hole pairs estimated in the leading order. The ray $Z = \sqrt{Y^\infty}X$ (dashed line) represents the dimer-atom threshold. The termination points along this threshold X_n^∞ are discussed in Eq. (10). Here $\beta = \sqrt{\gamma/\mu} = 10$, r_0 is the hard-core size of Bose atoms. Only Efimov states are shown, and we take $\delta = \pi/4$ [19].

for $-\infty < Y < Y^\infty$. This formula yields estimates for several quantities of interest. For example, from Eq. (8) we find that $N(X) \to \frac{\beta\Omega}{\pi} \log \frac{e^{[v(Y^\infty)]/\Omega}}{X} + \frac{\delta^\infty}{\pi}$ as X approaches 0. Furthermore, the termination points are $X_n^\infty \approx e^{(1/\Omega)\{[v(Y^\infty)]+(\delta^\infty/\beta)\}} e^{-(n\pi/\Omega\beta)}$ for small X_n^∞, which leads to a universal relation between different termination points:

$$X_n^\infty / X_{n+1}^\infty = e^{\pi/(\Omega\beta)}, \quad (10)$$

for small X_n^∞ or large n, and otherwise independent of n. We can also calculate $X_n(Y)$, the point at which the nth eigenvalue increases to Y, for $-\infty < Y < Y^\infty$; we obtain a result identical to Eq. (10) after replacing $X_n^\infty/X_{n+1}^\infty$ with $X_n(Y)/X_{n+1}(Y)$ in that expression. These are the new universal properties in three-body physics when there is a Fermi sea quantum background. Our numerical results agree well with the WKB results: for $\beta = 10$, the difference is within a few percent for the densities shown in Fig. 2.

These universal structures in the spectrum flow should be robust against the scattering of trimers by the Fermi sea. So far we have treated the Fermi sea as static or incompressible, but a trimer can further collide with the Fermi sea, exciting particle-hole pairs near the Fermi surface and polarizing the background. One of the main effects of these fluctuating pairs is to suppress the interspecies interactions, as occurs in an interacting Fermi gas [20], leading to a reduction of the Fermi-Bose dimer binding energy (estimated in Ref. [21]). Following those diagrammatic calculations, in the limit of heavy Bose atoms one finds that the dimer binding energy measured from the Fermi energy is reduced by a factor of m/M when $k_F a_{HL}$ is small and negative. This can also be attributed to an effect related to Anderson's infrared catastrophe [22]. As a result, Y^∞, which specifies the dimer-atom threshold for trimers, moves closer to unity (see Fig. 1). Y^∞ and $v(Y^\infty)$ are now functions of β, and these mass-ratio-dependent corrections to the effective potential shift the flow of each eigenvalue trajectory. However, they do not affect the universal relation between different trajectories, as Eq. (10) is independent of Y^∞ and $v(Y^\infty)$; we obtain the same universal value for $X_n^\infty/X_{n+1}^\infty$, provided X_n^∞ is small. At densities close to X_n^∞ and related by Eq. (10), we anticipate distinct peak-dip signatures in the inelastic loss spectrum similar to those observed in Refs. [8–13].

In conclusion, we have shown that the spectrum flow of Efimov states when the density of fermions varies is universal as far as the hard-core size of heavy Bose atoms r_0 is much shorter than the average interatomic distance. Our results can be applied to understanding three-body states in ^{87}Rb-^{6}Li and ^{23}Na-^{6}Li mixtures. It is possible to generalize the above idea to other situations. For instance, when applying the same approach to the case of one light boson in resonance with two heavy Fermi atoms in the presence of a Fermi sea, we find that because of the antiscreening effect of background Fermi atoms, the mediated potential $u_L(r) = -\Omega^2/r^2[1 - q_0 r \mathrm{si}(q_0 r)]$, where $q_0 = \beta^2 k_F/(2\pi)^2$ and $\mathrm{si}(x)$ is a sine integral which oscillates and approaches $-\frac{\cos x}{x}(1-\frac{2}{x^2})$ when $x \to \infty$. A similar spectrum flow can be obtained and details will be presented elsewhere. There are also open questions that need to be answered in the future. One is the role of holelike configurations and the effect of Fermi sea polarization on the three-body spectrum [23]. Another question is what are the dynamic consequences of these Efimov states embedded in a quantum gas and how do they affect the inelastic loss spectrum and dimer-atom or dimer-dimer elastic scattering [17]? The eventual fate of such a quantum gas after a large fraction of scattering atoms falls into the Efimov channel remains unknown. A straightforward way to proceed is perhaps to include more atoms in the Efimov channel and extend the analysis to 4-body, 5-body, etc. [6,7]. Along that road, quantum Monte Carlo simulations have been performed to understand cluster structures of bosonic atoms [24,25]. Similar simulations need to be performed in the presence of background scattering atoms. This work is supported by NSERC (Canada) and Canadian Institute for Advanced Research.

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Note added.—Recently, we learned that the polarization effect has been discussed in the context of an impurity problem [23].

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[18] The real function iS_1 represents the next order contribution (in terms of $1/\beta$) that we do not show explicitly. When r approaches zero, $\exp iS_1(r) \sim |u_L(r)|^{(1+\Omega-\Omega^2)/4(1-\Omega)}/|u_B(r) - u_L(r)|^{1/4}$.

[19] Since $u_L(r)$ oscillates around its asymptotic value, there can be multiple turning points when u_n is close to u_L^∞ and δ can depend on u_n. For instance, if there are additional turning points at r_2 and r_3, the phase shift is $\delta = \frac{\pi}{4} + \arctan(\frac{1}{4}\tan\theta e^{-2\eta})$ with $\theta = \beta\int_{r_2}^{r_3} dr'\sqrt{u_n - u_L(r')}$ and $\eta = \beta\int_{r_1}^{r_2} dr'|\sqrt{u_n - u_L(r')}|$. For typical values of u_n (i.e., θ is away from $\pi/2$) and large β, the deviation of δ from $\pi/4$ is exponentially small. However, for specific values of u_n, $\tan\theta$ can be large indicating additional non-Efimov states at these energies, bound in the oscillations at large $k_F r$. Very near these values, δ has a strong u_n dependence and Eq. (8) can have multiple solutions for a given n reflecting the emergence of non-Efimov bound states (not shown in Fig. 2). However, the hybridization between these and the Efimov states shown in Fig. 2 is exponentially suppressed in the large β limit. In this Letter, we only consider states in the Efimov family (which arise from the $k_F \to 0$ bound states); with this restriction Eq. (8) specifies a unique u_n for each n. These complications do not affect the universal structures discussed below.

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