Exact Time-Dependent Nonlinear Dispersive Wave Solutions in Compressible Magnetized Plasmas Exhibiting Collapse

Nikhil Chakrabarti and Chandan Maity
Saha Institute of Nuclear Physics, 1/AF Bidhannagar Calcutta–700 064, India

Hans Schamel
Theoretical Physics, University of Bayreuth, D-95440 Bayreuth, Germany

(Received 8 January 2011; published 8 April 2011)

Compressional waves in a magnetized plasma of arbitrary resistivity are treated with the Lagrangian fluid approach. An exact nonlinear solution with a nontrivial space and time dependence is obtained with boundary conditions as in Harris' current sheet. The solution shows competition among hydrodynamic convection, magnetic field diffusion, and dispersion. This results in a collapse of density and the magnetic field in the absence of dispersion. The dispersion effects arrest the collapse of density but not of the magnetic field. A possible application is in the early stage of magnetic star formation.

DOI: 10.1103/PhysRevLett.106.145003 PACS numbers: 52.30.Cv, 95.30.Qd

Singularities in hydrodynamics and other collective systems, described by continuum equations, such as plasmas in the fluid approximation, have acquired considerable attention recently due to the special role they play in the characterization and understanding of the underlying physical processes. As a rule, singularities appear in finite time, are purely nonlinear, i.e., they cannot be understood or predicted by a linearization of the governing equations, and are crucial events in the underlying physics describing changes in topology, i.e., are seeds for new structures requesting new physics.

A typical example in hydrodynamics is the drop formation and breakup of jets, where the neck undergoes zero shrinking at finite time [1]. Near the singular event a stochastic element in the governing Navier-Stokes equation has to be included to account for the droplet formation in agreement with the observations, as seen, e.g., in molecular dynamic simulations [2,3].

In plasma physics a comparable example is the plasma expansion into vacuum [4–6], which operates on the ionic time scale, such that electrons reside in an equilibrium state. During the evolution, described by ion fluid equations, the ion density experiences a collapse in finite time, which has been identified as an ion wave breaking scenario [6,7]. Beyond that, depending on the degree of collisionality, either kinetic or dissipative effects supervene, giving rise to the fast ion peak propagating supersonically into the vacuum [8–11], an experimentally well-established fact in laser-matter interactions [12].

In this Letter, we present another time-dependent, strongly nonlinear, collapsing solution, namely, one belonging to the realm of compressible MHD plasmas. The dynamical structure is spatially a localized current sheet, describing the reversal of the magnetic field across the sheet similar to a Harris sheet [13]. In contrast to the latter, however, the evolution is transient as both the magnetic field and the density are strongly time dependent, becoming singular in finite time simultaneously as long as dispersion is negligible. Dispersive effects are shown to prevent density from collapsing but not so for the magnetic field. It is argued that matter clumping in the Universe prior to star formation involving strong and ultrastrong magnetic fields may be seeded by such a process.

Our system of equations utilizes the geometry of compressional waves; i.e., the x and t dependent magnetic field is in the z direction, the electric field and current in the y direction, and the propagation and inhomogeneity of the structure in the x direction.

A shorthand approach to solving our system of equations is the use of resistive MHD equations supplemented by a generalized Ohm's law in which the $(j/n)^{\cdot}$ (where the dot implies the total time derivative) term is kept and in which an explicit use of the $1/n$ dependency of the resistivity is made. The magnetic field is assumed to be large such that the electron pressure contribution is negligible. Here we prefer a somewhat longer justification, as follows.

We consider a one-dimensional, cold, two-component, quasineutral ($n_e \sim n_i \sim n$) plasma; the ions are taken to be singly ionized without loss of generality. In the fluid approximation the basic equations describing the dispersive compressional wave in a magnetized plasma can be written as

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{2n}\frac{\partial B^2}{\partial x}, \tag{2}$$

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial x}(Bv) = \epsilon\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(\frac{1}{n}\frac{\partial B}{\partial x}\right) + \eta\frac{\partial}{\partial x}\left(\frac{1}{n}\frac{\partial B}{\partial x}\right). \tag{3}$$

The first two equations are the continuity and momentum equation for ions. To write Eq. (2), it is assumed that the magnetic pressure force is much greater than the thermal pressure force. Equation (3) is a combination of the electron momentum equation with Maxwell's equations. The term ϵ is the dispersion term, arising due to finite electron mass and η is the resistivity, arising due to electron-ion collisions. These equations are all in terms of dimensionless variables such that density is normalized by a constant value n_0 and the magnetic field is normalized by $\sqrt{4\pi n_0(m_e+m_i)v_A^2}$, where v_A is the Alfvén speed. The length scale is the arbitrary length L, and the time scale is the Alfvénic transit time v_A/L. The parameters ϵ and η are given by $\epsilon=(\delta/L)^2$, where $\delta = c\sqrt{m_e m_i/4\pi n_0 e^2(m_e+m_i)}$ and $\eta=\epsilon(m_e+m_i/m_i)\times(\nu_{ei}L/v_A)$.

Now we proceed to find an exact solution of Eqs. (1)–(3) using Lagrangian variables. In solving these equations we transform from Eulerian variables (x,t) to Lagrangian variables (ξ,τ) (such that $\xi=x$ at $t=0$) where $\tau\equiv t$ and $\xi\equiv x-\int_0^\tau d\tau' v(\xi,\tau')$, so that ξ is a function of both x and t, but ξ and τ are treated as independent variables. In terms of these new variables the convective derivative $\partial/\partial t+v\partial/\partial x$ becomes $\partial/\partial\tau$ and following previous work [14–20], we find from Eq. (1)

$$n(\xi,\tau)=n(\xi,0)\left[1+\int_0^\tau d\tau'\frac{\partial}{\partial\xi}v(\xi,\tau')\right]^{-1}\Rightarrow\frac{n(\xi,\tau)}{n(\xi,0)}=\frac{\partial\xi}{\partial x}, \tag{4}$$

where $n(\xi,0)$ represents the initial ($\tau=0$) density distribution in space and the corresponding fluid equations are

$$\frac{\partial^2}{\partial\tau^2}\left(\frac{1}{n}\right)=-\frac{1}{2n(\xi,0)}\frac{\partial}{\partial\xi}\left[\frac{1}{n(\xi,0)}\frac{\partial B^2}{\partial\xi}\right], \tag{5}$$

$$\frac{\partial}{\partial\tau}\left(\frac{B}{n}\right)=\frac{1}{n(\xi,0)}\frac{\partial}{\partial\xi}\left[\frac{1}{n(\xi,0)}\frac{\partial}{\partial\xi}\left(\epsilon\frac{\partial B}{\partial\tau}+\eta B\right)\right]. \tag{6}$$

To derive Eqs. (5) and (6) we have combined Eqs. (1), (2) and (1), (3) after converting them to Lagrangian variables. These equations tell us that in the absence of dispersions ($\epsilon=0$) and dissipation ($\eta=0$) the magnetic field is frozen in the plasma and that a finite value of either of these parameter breaks that symmetry. We shall now present the solutions, which we obtained by the method of separation of variables. Proposing the solutions of the form $n(\xi,\tau)=N(\xi)\phi(\tau)$, $B(\xi,\tau)=b(\xi)\psi(\tau)$, one can substitute in Eqs. (5) and (6) and separate space and time variable equations are as follows:

$$-\frac{\phi^2(0)}{\psi^2}\frac{d^2}{d\tau^2}\left(\frac{1}{\phi}\right)=\frac{1}{2}\frac{d}{d\xi}\left[\frac{1}{N(\xi)}\frac{db^2}{d\xi}\right]=\alpha, \tag{7}$$

$$\frac{\phi^2(0)}{\epsilon\dot\psi+\eta\psi}\frac{d}{d\tau}\left(\frac{\psi}{\phi}\right)=\frac{1}{b}\frac{d}{d\xi}\left[\frac{1}{N(\xi)}\frac{db}{d\xi}\right]=\beta, \tag{8}$$

where $n(\xi,0)=N(\xi)\phi(0)$, $B(\xi,0)=b(\xi)\psi(0)$ with $\psi(0)$, $\phi(0)\neq 0$ and α, β being arbitrary separation constants. Here the overdot on ψ in Eq. (8) implies derivative with respect to time τ. Solving separately for spatial and temporal equations one can obtain a complete solution [21]. Note here that in the temporal solution an auxiliary variable θ is introduced in place of time such that

$$\frac{1}{\psi}\frac{d}{d\tau}=f(\theta)\frac{d}{d\theta}=\exp\left(\frac{\eta\beta\theta}{\beta\epsilon-1}\right)\frac{d}{d\theta},$$

where $f(\theta)$ plays the role of an integrating factor.

The parameter α stands for the strength of the magnetic field and controls the time scale [21]. If it is, without loss of generality, chosen to be 2 and if the initial density is normalized to unity, yielding $\beta=\pi/2$, we get for density and magnetic field

$$n(\xi,\tau)=\frac{1}{\beta}\left\{\frac{1}{(1+\xi^2)^2}\right\}\times\left[\frac{1}{\beta\epsilon+(1-\beta\epsilon)\cos^2\tilde\theta\exp(2\tilde\eta\,\tilde\theta)}\right], \tag{9}$$

$$B(\xi,\tau)=\sqrt{\frac{2}{\beta}}\left\{\frac{\xi}{\sqrt{1+\xi^2}}\right\}\frac{1}{\cos^2\tilde\theta}\exp(-\tilde\eta\,\tilde\theta), \tag{10}$$

where $\tilde\theta=\theta/\sqrt{1-\beta\epsilon}$ and $\tilde\eta=\eta\beta/\sqrt{1-\beta\epsilon}$. Moreover, the mean velocity $v(\xi,\tau)$, which can be obtained by a ξ integration of the continuity equation [21], is found to be $v(\xi,\tau)=2\sqrt{1-\beta\epsilon}\xi[\tilde\eta-\tan\tilde\theta]$. Once the solution for the density is known one can easily find out, by utilizing (4), the relation between ξ and x: $x=\xi[\beta\epsilon+(1-\beta\epsilon)\cos^2\tilde\theta\exp(2\tilde\eta\,\tilde\theta)]$, which is linear in space but strongly time dependent.

The relation between τ and θ is given by

$$\tau=\frac{\sqrt{1-\beta\epsilon}}{4}\left[e^{2\tilde\eta\,\tilde\theta}\left(\frac{\sin2\tilde\theta+\tilde\eta\cos2\tilde\theta}{1+\tilde\eta^2}\right)-\frac{\tilde\eta}{1+\tilde\eta^2}+\frac{1}{\tilde\eta}(e^{2\tilde\eta\,\tilde\theta}-1)\right]. \tag{11}$$

Equations (9)–(11) represent the complete solution, which depends on the two independent, external parameters ϵ and η. At $\tau=0$ ($\theta=0$, respectively) it resembles the Harris sheetlike solution, $B\sim\tanh\zeta$, $n\sim\mathrm{sech}^4\zeta$, seen by setting $\xi\sim\sinh\zeta$, but, in contrast to it, evolves now strongly time dependent even in a cold plasma environment.

To see its consequences, we first analyze the dispersionless, ideal limit by setting $\epsilon=0=\eta$, and get

$$
\begin{aligned}
n(\xi,\tau) &= \frac{1}{\beta}\frac{1}{(1+\xi^2)^2}\frac{1}{\cos^2\theta};\\
B(\xi,\tau) &= \sqrt{\frac{2}{\beta}}\frac{\xi}{\sqrt{1+\xi^2}}\frac{1}{\cos^2\theta};\\
v(\xi,\tau) &= -2\xi\tan\theta;\\
x &= \xi\cos^2\theta;\\
\tau &= \frac{1}{4}(2\theta+\sin 2\theta).
\end{aligned}
\tag{12}
$$

Starting from a Harris-type initial state there is an inward drift of the fluid, blowing up the density and magnetic field and shrinking the inhomogeneity width. At $\theta_c = \pi/2 = 1.571$ ($t_c = \tau_c = \pi/4 = 0.785$, respectively) the system collapses giving rise to a singularity in n, B and v at the collapse point $x_c = 0$.

Compressibility, nonlinearity, and a Lorentz-force driven convection result in an unbounded amplification in finite time, as long as dispersion and dissipation are absent.

For nonvanishing dispersion and resistivity, the full set (9)–(11) applies, in which $\tilde{\eta}\,\tilde{\theta}$ is given by $\beta\eta\theta/(1-\beta\epsilon)$. As long as $\beta\epsilon < 1$ the denominator of the density in (9), however, can no longer become zero even in the zero resistivity limit. The action of the dispersion alone prevents the density from collapsing, whereas the magnetic field still experiences collapse, namely, at $\theta_c = \sqrt{(1-\beta\epsilon)}\pi/2$, i.e., earlier in the auxiliary time variable. There is still a collapsing inward drift, as seen from the $v(\xi,\tau)$ expression, but finite resistivity delays the drift motion by adding an outward drift component. At collapse, when $\cos^2\tilde{\theta} = 0$, the spatial width in x remains finite in accordance with the limitation of the density.

Figure 1 displays the space-time behavior of the magnetic field B for finite values of the dispersion, $\epsilon = 0.1$, and the resistivity, $\eta = 0.03$. The collapse is clearly seen by the blowing up of B in time just before $t_c = 0.757$, measured in terms of Alfvén transit time L/v_A.

This is in contrast to the density behavior, plotted in Fig. 2 for the same set of parameters. Close to the magnetic field collapse event the density becomes stronger peaked and narrower, but remains finite.

The relationship between τ and θ is plotted in Fig. 3 for three cases: (i) $\epsilon = 0 = \eta$ (solid line); (ii) $\epsilon = 0.1$, $\eta = 0$ (dotted line); and (iii) $\epsilon = 0.1$, $\eta = 0.03$ (dash-dotted line). We recognize that dispersion and resistivity act oppositely. Whereas dispersion speeds up the collapse process, resistivity delays it.

This is also seen from the relationship between θ_c and the actual time $\tau_c = t_c$ at magnetic field collapse ($\tilde{\theta}_c = \pi/2$) in the small resistivity limit $\tilde{\eta} \ll 1$:

$$
\begin{aligned}
\tau_c &= \frac{\sqrt{1-\beta\epsilon}}{4\tilde{\eta}(1+\tilde{\eta}^2)}[\exp(\tilde{\eta}\pi) - 1 - 2\tilde{\eta}^2]\\
&\simeq \frac{\sqrt{1-\beta\epsilon}}{4}[\pi + (\pi^2/2 - 2)\tilde{\eta} + \cdots].
\end{aligned}
\tag{13}
$$

Notice that in the dissipationless case, $\eta = 0$, the collapse time can also be obtained through $\tau'(\tilde{\theta}_c) = 0$.

We conclude that the initial configuration of a Harris-type current sheet collapses together with the associated magnetic field, when subject to the fully nonlinear, time-dependent, generalized MHD equations, represented by (1)–(3). The density collapses as well unless dispersive effects enter the system. Resistivity has been found not to be able to halt the density collapse; rather, it merely has a delaying effect on the process.

The solution of this problem was made possible by the introduction of the auxiliary time variable θ instead of τ through which the complex time dependency could be transformed and simplified.

For arbitrary initial magnetic field strength $\sqrt{\alpha}$ [21] the collapse time is modified becoming $\theta_c = \pi\sqrt{(1-\beta\epsilon)/2\alpha}$; i.e., it is prolonged if the field is weaker, as expected.

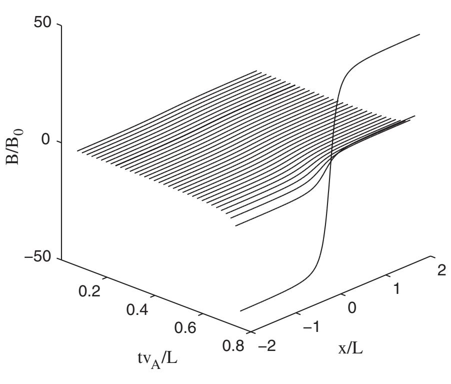

FIG. 1. Normalized magnetic field evolution with finite dispersion $\epsilon = 0.1$ and dissipation $\eta = 0.03$. The figure shows that neither finite dispersion nor resistivity can stop the magnetic field singularity. The latter has merely a delaying effect on the collapse time, which occurs at $t_c = 0.757$, the time measured in Alfvén transit time.

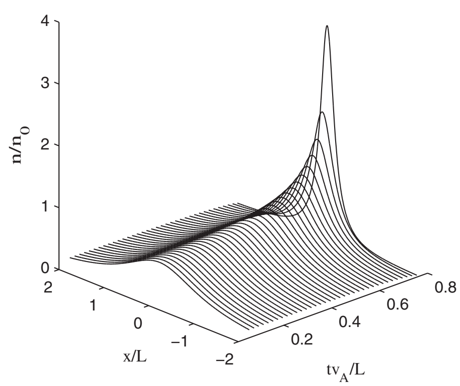

FIG. 2. Normalized density evolution with finite dispersion $\epsilon = 0.1$ and resistivity $\eta = 0.03$. The figure shows that the density singularity is removed due to finite dispersion.

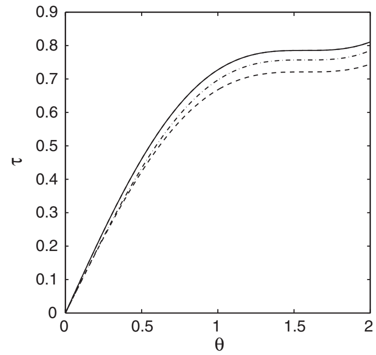

FIG. 3. Variation of the actual time τ with respect to the timelike variable θ for three cases: (i) $\epsilon = 0 = \eta$; $t_c = 0.785$ (solid line); (ii) $\epsilon = 0.1$, $\eta = 0$; $t_c = 0.721$ (dotted line); and (iii) $\epsilon = 0.1$, $\eta = 0.03$; $t_c = 0.757$ (dash-dotted line).

As mentioned in the introduction, a singular behavior indicates the appearance of new physics. Candidates in the present case are effects arising from (electron) pressure and viscosity, but also relativistic or gyrokinetic (and even kinetic) effects can enter. Moreover, the one-dimensional (1D) structure will probably be unstable against transversal perturbations such that 3D localized patterns of high density and strong magnetic field are expected to arise. It would, therefore, be no surprise if this compressional-type collapse process driven initially by a localized current sheet turns out to be the seed for star formation prior to the onset of gravitation and turbulence especially when strong magnetic fields are involved.

Indeed, the observed preservation of magnetic field orientation in the star formation [22] during the accumulation of a low-density, large scale intercloud medium to a high-density, small scale cloud core indicates that field tangling due to turbulent eddies cannot be severe at least for strong fields (sub-Alfvénic turbulence) such that the magnetic field itself must be dynamically significant in comparison to the mass accumulation agents, i.e., gravity and turbulence. Our model provides a clue for the understanding of such a star formation process, which is dominated by magnetic fields.

In conclusion, we emphasize that our analysis of a compressional MHD kind collapse has been oriented towards a simple macroscopic situation in which nonlinearity, time dependence, dispersion, and resistive dissipation are treated on equal footing, resulting in an exact solution of the governing equations. We could show that the collapse driven by a localized Harris-type current sheet configuration affects both density and magnetic field in an ideal, dispersionless plasma situation. Whereas resistivity cannot stop collapse and merely delays its appearance, dispersion is able to stop the density from collapsing.

This type of solution represents a new class of nonlinear transient solutions that may arise in a manifold of similar physical situations. Exact nonlinear, time-dependent solutions involving dispersive and dissipative effects are rare in the literature and can probably be achieved through the Lagrangian fluid description only. Another physical system, which could be attacked solely by this procedure, is the beam transport in plasma diodes under the influence of collisions. The latter modifies the space-charge-limited current, building a bridge between the ballistic Child-Langmuir law and the nonballistic drift Mott-Gurney law [23]. We believe that more investigations in this direction will enrich our understanding of such intricate physical processes such as star formation.

N. C. would like to acknowledge the discussions with Professor P. K. Kaw, Dr. S. Sengupta, and R. Loganayagam.

[1] J. Eggers, Rev. Mod. Phys. **69**, 865 (1997); Phys. Rev. Lett. **89**, 084502 (2002).
[2] M. Moseler and U. Landman, Science **289**, 1165 (2000).
[3] R. W. Batterman, Stud. Hist. Phil. Mod. Phys. **36**, 225 (2005).
[4] A. V. Gurevich, L. V. Pariiskaya, and L. P. Pitaevskii, Sov. Phys. JETP **22**, 449 (1966).
[5] J. E. Crow, P. L. Auer, and J. E. Allen, J. Plasma Phys. **14**, 65 (1975).
[6] C. Sack and H. Schamel, J. Comput. Phys. **53**, 395 (1984); Plasma Phys. Controlled Fusion **27**, 717 (1985).
[7] C. Sack and H. Schamel, Phys. Rep. **156**, 311 (1987).
[8] A. E. Kaplan, B. Y. Dubetsky, and P. L. Shkolnikov, Phys. Rev. Lett. **91**, 143401 (2003).
[9] T. Grismayer and P. Mora, Phys. Plasmas **13**, 032103 (2006).
[10] F. Peano *et al.*, Phys. Plasmas **14**, 056704 (2007).
[11] R. Shokoohi and H. Abbasi, J. Appl. Phys. **106**, 033309 (2009).
[12] H. Schwoerer *et al.*, Nature (London) **439**, 445 (2006).
[13] E. Harris, Nuovo Cimento **23**, 115 (1962).
[14] G. Kalman, Ann. Phys. (N.Y.) **10**, 1 (1960).
[15] P. A. Sturrock, Proc. R. Soc. A **242**, 277 (1957); Phys. Rev. **121**, 18 (1961).
[16] R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic, New York, 1972).
[17] H. Schamel, Phys. Rep. **392**, 279 (2004).
[18] G. Rowlands, G. Brodin, and L. Stenflo, J. Plasma Phys. **74**, 569 (2007).
[19] N. Chakrabarti and M. S. Janaki, Phys. Plasmas **10**, 3043 (2003).
[20] I. Kourakis and P. K. Shukla, Eur. Phys. J. D **30**, 97 (2004).
[21] N. Chakrabarti, C. Maity, and H. Schamel (to be published).
[22] H. bai Li *et al.*, Astrophys. J. **704**, 891 (2009); Mon. Not. R. Astron. Soc. **411**, 2067 (2011).
[23] P. Akimov and H. Schamel, J. Appl. Phys. **1690**, 2002 (2002).