Entanglement Cost in Practical Scenarios

Francesco Buscemi[1,*] and Nilanjana Datta[2,†]
[1]Institute for Advanced Research, Nagoya University, Nagoya 464-8601, Japan
[2]Statistical Laboratory, DPMMS, University of Cambridge, Cambridge CB3 0WB, United Kingdom
(Received 22 September 2010; published 30 March 2011)

We quantify the one-shot entanglement cost of an arbitrary bipartite state, that is, the minimum number of singlets needed by two distant parties to create a single copy of the state up to a finite accuracy, by using local operations and classical communication only. This analysis, in contrast to the traditional one, pertains to scenarios of practical relevance, in which resources are finite and transformations can be achieved only approximately. Moreover, it unveils a fundamental relation between two well-known entanglement measures, namely, the Schmidt number and the entanglement of formation. Using this relation, we are able to recover the usual expression of the entanglement cost as a special case.

DOI: 10.1103/PhysRevLett.106.130503 PACS numbers: 03.67.Mn

Among quantum-information processing tasks, entanglement manipulation, namely, the interconversion between entangled states using only local transformations and classical communication, represents an important primitive. In this scenario, the abstract notion of entanglement becomes a fungible resource ''as real as energy'' [1]. This is one of the reasons for which intensive research has been devoted to the study of entanglement manipulations since the very early stages of Qquantum-information theory, making such an operational theory of entanglement one of its biggest successes.

In this context, however, the word ''operational'' should not be confused with ''practical.'' Indeed, most results we have at present about entanglement resource theory rely on two unrealistic (and very strong) assumptions: (i) Many independent and identically distributed (i.i.d.) copies of the initial resource (e.g., the initial entangled state) are to be converted into many i.i.d. copies of the target state. This corresponds to assuming the absence of correlations in the noisy (partially entangled) states which are either produced or consumed by the entanglement manipulation procedure. (ii) The optimal interconversion rate is computed as the asymptotic input-to-output ratio, in the limit of infinitely many initial and final copies.

These two assumptions constitute what is usually called the asymptotic i.i.d. scenario. In order to establish a truly general entanglement resource theory, then, one should drop both assumptions (i) and (ii). The highest possible degree of theoretical generality is described by the so-called one-shot scenario, in which a single initial state has to be transformed into a single desired final state, up to a finite accuracy. Incidentally, this is indeed the scenario in which experiments are performed, since resources available in nature are typically finite and correlated, and transformations can be achieved only approximately.

One end of such a generalized entanglement resource theory, namely, one-shot entanglement distillation, was considered by the present authors in Ref. [2]: There we described the case of two distant parties trying to convert, up to some fixed error ε, a finite number of initially shared noisy bipartite entangled states into noiseless entanglement, i.e., singlets, by using local operations and classical communication (LOCC) only. In this Letter, we completely characterize the other end of the theory, namely, one-shot entanglement dilution: Here the goal is to utilize a finite amount of initial noiseless entanglement to produce (again, by LOCC and up to some fixed error ε) a single bipartite target state ρ_{AB}, which might not be directly available otherwise. In this scenario, entanglement dilution is relevant as the ''reverse'' of entanglement distillation: It shows that singlets indeed provide a universal resource from which any bipartite state can be obtained by LOCC, quantifying, at the same time, the minimum amount of singlets needed (i.e., the cost) to produce a given bipartite state.

Our main result [3] is a formula for the minimum number of singlets necessary for successfully producing a given target state ρ_{AB} up to a finite error ε. We refer to this quantity as the one-shot entanglement cost $E_C^{(1)}(\rho_{AB};\varepsilon)$. The formula we derive involves a generalized quantum relative entropy, namely, the relative Rényi entropy of order zero [4], and makes use of a smoothing procedure similar to that introduced in Ref. [5]. When specialized to the asymptotic i.i.d. scenario, our formula yields the entanglement cost given in terms of the regularized entanglement of formation [6,7]. This is in accordance with the claim that the one-shot entanglement resource theory is more general than the asymptotic i.i.d. one. Finally, as a by-product of our findings, we are able to prove that two entanglement monotones, namely, the entanglement of formation [6] and the Schmidt number [8], which were previously considered to be unrelated, are in fact directly connected, in the sense that the former is recovered from the latter by suitable smoothing and regularization, as explained below.

Basic concepts.—In order to clearly state our main results, given in Theorems 1 and 2 below, we first have to introduce some notations and definitions. Throughout the paper, the letter $\mathcal{H}$ denotes finite-dimensional Hilbert space, whereas $\mathfrak{S}(\mathcal{H})$ denotes the set of states (or density operators, i.e., positive operators of unit trace) acting on $\mathcal{H}$. Furthermore, let $\mathbb{1}$ denote the identity operator acting on $\mathcal{H}$. Given a positive operator $\omega \geq 0$, we denote by Π_ω the projector onto its support, and, for a pure state $|\varphi\rangle$, we denote the projector $|\varphi\rangle\langle\varphi|$ simply as φ. Moreover, given two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, of dimensions d_A and d_B, respectively, with two given orthonormal bases $\{|i_A\rangle\}_{i=1}^{d_A}$ and $\{|i_B\rangle\}_{i=1}^{d_B}$, we define the canonical maximally entangled state in $\mathcal{H}_A \otimes \mathcal{H}_B$ of Schmidt number $M \leq \min\{d_A, d_B\}$ to be $|\Psi_M^+\rangle = M^{-1/2}\sum_{i=1}^{M}|i_A\rangle \otimes |i_B\rangle$.

Information-theoretical protocols, since Shannon, are usually characterized in terms of suitable entropic quantities. In quantum-information theory, too, entropic quantities like the von Neumann entropy, the conditional entropy, and the mutual information are often encountered. All these quantities can in fact be derived from the quantum relative entropy [4], which is defined, for a state ρ and an operator $\sigma \geq 0$, as

$$S_r(\rho \parallel \sigma) := \begin{cases} \mathrm{Tr}[\rho\log\rho - \rho\log\sigma], & \text{if } \Pi_\rho \leq \Pi_\sigma, \\ +\infty, & \text{otherwise.} \end{cases}$$

(The logarithm in the above equation and in what follows is taken to base 2.) For example, the von Neumann entropy of a state ρ, defined as $S(\rho) := -\mathrm{Tr}[\rho\log\rho]$, can be equivalently written as $S(\rho) = -S_r(\rho \parallel \mathbb{1})$. Our main results are, however, expressed in terms of an alternative relative entropy, namely, the relative Rényi entropy of order zero, which, for a state ρ and an operator $\sigma \geq 0$, is defined as

$$S_0(\rho \parallel \sigma) := \begin{cases} -\log\mathrm{Tr}[\Pi_\rho\sigma], & \text{if } \mathrm{Tr}[\Pi_\rho\Pi_\sigma] \neq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

From these two relative entropies S_r and S_0, we define the corresponding conditional entropy of a given bipartite state ρ_{AB} given a state σ_B as

$$H_\star(\rho_{AB}|\sigma_B) := -S_\star(\rho_{AB} \parallel \mathbb{1}_A \otimes \sigma_B) \quad (1)$$

and the conditional entropy of ρ_{AB} given the subsystem B as

$$H_\star(\rho_{AB}|B) := \max_{\sigma_B\in\mathfrak{S}(\mathcal{H}_B)} H_\star(\rho_{AB}|\sigma_B), \quad (2)$$

for $\star \in \{r, 0\}$. It turns out (see, e.g., Lemma 6 in [9]) that $H_r(\rho_{AB}|B) = H_r(\rho_{AB}|\rho_B) = S(\rho_{AB}) - S(\rho_B)$, where $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$, for any given ρ_{AB}. However, in general, $H_0(\rho_{AB}|B) \neq H_0(\rho_{AB}|\rho_B)$.

It is also convenient to introduce, for any given decomposition of a bipartite state ρ_{AB} into a pure-state ensemble $\mathfrak{E} = \{p_i, |\phi_{AB}^i\rangle\}$ such that $\sum_i p_i\phi_{AB}^i = \rho_{AB}$, the tripartite classical-quantum (c-q) state

$$\rho_{RAB}^{\mathfrak{E}} := \sum_i p_i|i\rangle\langle i|_R \otimes \phi_{AB}^i, \quad (3)$$

where R denotes an auxiliary classical system represented by the fixed orthonormal basis $\{|i_R\rangle\}$. Given a pure-state ensemble $\mathfrak{E}$, let $\rho_A^i := \mathrm{Tr}_B[\phi_{AB}^i]$, for all i.

As noted earlier, in the realistic scenario of finite entanglement resources and imperfect transformations, one is compelled to allow for a nonvanishing error, say, ε, in achieving the final desired state. This error ε manifests itself as a "smoothing" of the underlying information-theoretical quantity characterizing the task, which in our case turns out to be a conditional Rényi entropy of order zero. This fact leads us to define, in analogy with Ref. [5], a smoothing as follows: For any $\varepsilon \geq 0$ and any pure-state ensemble $\mathfrak{E} = \{p_i, |\phi_{AB}^i\rangle\}$ of ρ_{AB}, we define the c-q–smoothed conditional zero-Rényi entropy of the c-q state $\rho_{RA}^{\mathfrak{E}} := \mathrm{Tr}_B[\rho_{RAB}^{\mathfrak{E}}] = \sum_i p_i|i\rangle\langle i|_R \otimes \rho_A^i$, given R, as

$$H_0^\varepsilon(\rho_{RA}^{\mathfrak{E}}|R) := \min_{\omega_{RA}\in B_{\mathrm{cq}}^\varepsilon(\rho_{RA}^{\mathfrak{E}})} H_0(\omega_{RA}|R), \quad (4)$$

where the minimum is taken over classical-quantum operators belonging to the set $B_{\mathrm{cq}}^\varepsilon(\rho_{RA}^{\mathfrak{E}})$ defined, for any pure-state ensemble $\mathfrak{E} = \{p_i, |\phi_{AB}^i\rangle\}$ of ρ_{AB}, as follows:

$$B_{\mathrm{cq}}^\varepsilon(\rho_{RA}^{\mathfrak{E}}) := \Big\{\omega_{RA} \geq 0 | \omega_{RA} = \sum_i |i\rangle\langle i|_R \otimes \omega_A^i \text{ and } \|\omega_{RA} - \rho_{RA}^{\mathfrak{E}}\|_1 \leq \varepsilon\Big\},$$

with $\|X\|_1 := \mathrm{Tr}|X|$. The basis $\{|i_R\rangle\}$ used in the above definition is the same as that appearing in Eq. (3). Note that operators in $B_{\mathrm{cq}}^\varepsilon(\rho_{RA}^{\mathfrak{E}})$ are actually very close to being density operators, since $1 - \varepsilon \leq \mathrm{Tr}[\omega_{RA}] \leq 1 + \varepsilon$, for any $\omega_{RA} \in B_{\mathrm{cq}}^\varepsilon(\rho_{RA}^{\mathfrak{E}})$.

Main result.—Two parties, Alice and Bob, share a single copy of a maximally entangled state $|\Psi_M^+\rangle$ of Schmidt number M and wish to convert it into a given bipartite target state ρ_{AB} by using an LOCC map Λ. We refer to the protocol used for this conversion as one-shot entanglement dilution. For the sake of generality, we consider the situation where the final state of the protocol is ε-close to the target state with respect to a suitable distance measure, for any given $\varepsilon \geq 0$. As a measure of closeness, we choose here the (squared) fidelity, which is defined, for states ρ and σ, as $F^2(\rho, \sigma) := (\mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}|)^2$. In this way, defining the fidelity of the protocol to be $F^2(\Lambda(\Psi_M^+), \rho_{AB})$, we require $F^2(\Lambda(\Psi_M^+), \rho_{AB}) \geq 1 - \varepsilon$. Furthermore, for any given initial resource $|\Psi_M^+\rangle$ and any given target state ρ_{AB}, we denote the optimal fidelity of one-shot entanglement dilution as

$$\mathsf{F}_{\mathrm{dil}}(\rho_{AB}, M) := \max_{\Lambda\in\mathrm{LOCC}} F^2(\Lambda(\Psi_M^+), \rho_{AB}).$$

Definition 1 (one-shot entanglement cost).—For any given ρ_{AB} and $\varepsilon \geq 0$, the one-shot entanglement cost is defined as follows:

$$E_C^{(1)}(\rho_{AB}; \varepsilon) := \min_{M \in \mathbb{N}} \{\log M \colon \mathsf{F}_{\mathrm{dil}}(\rho_{AB}, M) \geq 1 - \varepsilon\}.$$

Notice that, by its very definition, the one-shot entanglement cost $E_C^{(1)}(\rho_{AB}; \varepsilon)$ constitutes, for any $\varepsilon \geq 0$, an entanglement (weak) monotone, in that it cannot increase under the action of an LOCC map [10]. As mentioned earlier, the smoothing here emerges naturally from a purely operational consideration, in the sense that it is a natural consequence of the finite accuracy we allow in the protocol. This is in contrast to the approach adopted in Ref. [11], where a smoothing is instead introduced axiomatically.

Our main result is given by the following theorem:

Theorem 1.—For any given target state ρ_{AB} and any given error parameter $\varepsilon \geq 0$, the one-shot entanglement cost under LOCC, corresponding to an error less than or equal to ε, satisfies the following bounds:

$$\min_{\mathfrak{E}} H_0^{2\sqrt{\varepsilon}}(\rho_{RA}^{\mathfrak{E}}|R) \leq E_C^{(1)}(\rho_{AB}; \varepsilon) \leq \min_{\mathfrak{E}} H_0^{\varepsilon/2}(\rho_{RA}^{\mathfrak{E}}|R),$$

where the minimum is taken over all pure-state ensemble decompositions $\mathfrak{E} = \{p_i, |\phi_{AB}^i\rangle\}$ of ρ_{AB}, and $\rho_{RA}^{\mathfrak{E}} = \mathrm{Tr}_B[\rho_{RAB}^{\mathfrak{E}}]$, with $\rho_{RAB}^{\mathfrak{E}}$ being the tripartite extension of ρ_{AB} defined in (3).

For any given $\varepsilon \geq 0$, Theorem 1 essentially identifies $\min_{\mathfrak{E}} H_0^{\varepsilon}(\rho_{RA}^{\mathfrak{E}}|R)$ as the quantity representing the one-shot entanglement cost $E_C^{(1)}(\rho_{AB}; \varepsilon)$ [12].

The theory developed here not only provides a complete characterization of the one-shot entanglement cost, it also yields a simple proof of a fundamental asymptotic result. It is known [7] that the asymptotic entanglement cost $E_C(\rho_{AB})$ of preparing a bipartite state ρ_{AB} is equal to the regularized entanglement of formation, defined as

$$E_F^{\infty}(\rho_{AB}) := \lim_{n \to \infty} \frac{1}{n} E_F(\rho_{AB}^{\otimes n}), \tag{5}$$

where $E_F(\rho_{AB}) := \min_{\mathfrak{E}} \sum_i p_i S(\rho_A^i)$ denotes the entanglement of formation of the state ρ_{AB} [6]. Applying our main result, Theorem 1, to the case of multiple (n) copies of the bipartite state ρ_{AB} and taking the asymptotic limit ($n \to \infty$) yields a new proof of the identity $E_C(\rho_{AB}) = E_F^{\infty}(\rho_{AB})$:

Theorem 2.—For any given target state ρ_{AB}, the following identity holds:

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} E_C^{(1)}(\rho_{AB}^{\otimes n}; \varepsilon) = E_F^{\infty}(\rho_{AB}). \tag{6}$$

Theorem 2, together with the results in Ref. [7], establishes that the asymptotic entanglement cost is alternatively expressible as the regularized one-shot entanglement cost, in the limit $\varepsilon \to 0^+$.

The theorems stated above emphasize the generality and twofold relevance of the one-shot analysis: On one hand, it gives a complete description of realistic scenarios of entanglement dilution; on the other hand, it provides a unified theoretical framework from which previous results can be derived as special cases.

Discussion.—In the case of perfect (zero-error) entanglement dilution, corresponding to the case $\varepsilon = 0$, Theorem 1 says that the corresponding one-shot entanglement cost is given by

$$E_C^{(1)}(\rho_{AB}; 0) = \min_{\mathfrak{E}} H_0(\rho_{RA}^{\mathfrak{E}}|R). \tag{7}$$

The above equation can be made more explicit as follows:

$$E_C^{(1)}(\rho_{AB}; 0) = \min_{\mathfrak{E}} \max_i \log \mathrm{Tr}[\Pi_{\rho_A^i}],$$

where, for any given pure-state ensemble decomposition $\mathfrak{E} = \{p_i, |\phi_{AB}^i\rangle\}$ of ρ_{AB}, $\rho_A^i := \mathrm{Tr}_B[\phi_{AB}^i]$. The quantity on the right-hand side of the equation above coincides with the logarithm of the Schmidt number (LSN) of the mixed state ρ_{AB}, introduced and studied in Ref. [8]. In Ref. [13], the same quantity was denoted as $E_{sr}(\rho_{AB})$ and was shown to characterize the zero-error entanglement cost $E_C^{(1)}(\rho_{AB}; 0)$. However, until now, there was a gap in the theory of entanglement dilution, in the sense that it was unclear how these zero-error results could be related to the usual notion of entanglement cost, for which the error vanishes only in the asymptotic limit.

The results we presented above show that it is indeed possible to fill such a gap by suitably smoothing the zero-error quantities. In fact, let us introduce a smoothed LSN as follows:

$$E_{\mathrm{sr}}^{\varepsilon}(\rho_{AB}) := \min_{\omega_{AB} \in C_{\varepsilon}(\rho_{AB})} E_{\mathrm{sr}}(\omega_{AB}), \tag{8}$$

where now the smoothing is performed with respect to the compact set of normalized states $C_{\varepsilon}(\rho_{AB})$ centered at ρ_{AB} defined as

$$C_{\varepsilon}(\rho_{AB}) := \{\omega_{AB} \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B) | F^2(\omega_{AB}, \rho_{AB}) \geq 1 - \varepsilon\}.$$

Then, using the arguments given below, one can prove that, for any $\varepsilon \geq 0$, the identity $E_C^{(1)}(\rho_{AB}; \varepsilon) = E_{\mathrm{sr}}^{\varepsilon}(\rho_{AB})$ holds. First, for any $\omega_{AB} \in C_{\varepsilon}(\rho_{AB})$, $E_{\mathrm{sr}}(\omega_{AB})$ singlets can be used to create, with zero error, the state ω_{AB}, which is, by construction, ε-close to ρ_{AB}. This proves that $E_C^{(1)}(\rho_{AB}; \varepsilon) \leq E_{\mathrm{sr}}^{\varepsilon}(\rho_{AB})$. For the other direction, let us assume that $E_C^{(1)}(\rho_{AB}; \varepsilon) < E_{\mathrm{sr}}^{\varepsilon}(\rho_{AB})$. Definition 1 then implies that, with $E_C^{(1)}(\rho_{AB}; \varepsilon)$ singlets, it is possible to create a state, say, $\tilde{\omega}_{AB}$, which is ε-close to ρ_{AB}. This in turn implies that $\tilde{\omega}_{AB} \in C_{\varepsilon}(\rho_{AB})$, with $E_{\mathrm{sr}}(\tilde{\omega}_{AB}) = E_C^{(1)}(\rho_{AB}; \varepsilon) < E_{\mathrm{sr}}^{\varepsilon}(\rho_{AB})$, which contradicts the fact that $E_{\mathrm{sr}}^{\varepsilon}(\rho_{AB})$ is defined as a minimum in (8).

We hence obtain the following corollary of Theorem 2:

Corollary 1.—For any given state ρ_{AB}, the entanglement of formation $E_F(\rho_{AB})$ and the LSN $E_{sr}(\rho_{AB})$ are related as follows:

$$\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\frac{1}{n}E^{\varepsilon}_{sr}(\rho_{AB}^{\otimes n}) = E_F^{\infty}(\rho_{AB}). \tag{9}$$

Essence of proofs.—We present here only the main steps of the proofs of the results stated above. The interested reader is referred to Ref. [3] for detailed derivations. The proof of Theorem 1 relies on the following lemma:

Lemma 1 [13,14].—For any given bipartite state ρ_{AB}, the optimal dilution fidelity is given by

$$\mathsf{F}_{dil}(\rho_{AB}, M) = \max_{\mathfrak{E}}\sum_i p_i \sum_{j=1}^{M}\lambda_j^{(i)}, \tag{10}$$

where the maximum is over all pure-state decomposition $\mathfrak{E} = \{p_i, |\phi^i_{AB}\rangle\}$ of ρ_{AB}, and $\{\lambda_j^{(i)}\}_j$ are the eigenvalues of $\rho^i_A = \mathrm{Tr}_B[\phi^i_{AB}]$, arranged in nonincreasing order.

Using this lemma and Definition 1, we can prove that $E_C^{(1)}(\rho_{AB};\varepsilon) = \min_{\mathfrak{E}} E^{\varepsilon}(\mathfrak{E})$, where

$$E^{\varepsilon}(\mathfrak{E}) := \min_{\{\Pi^i_A\}}\left\{\max_i \log\mathrm{Tr}[\Pi^i_A]\,\middle|\,\sum_i p_i\mathrm{Tr}[\Pi^i_A\rho^i_A]\geq 1-\varepsilon\right\},$$

where $\{\Pi^i_A\}$ is an unconstrained set of projectors, that is, not necessarily orthogonal nor complete. The proof of Theorem 1 then reduces to proving that $H_0^{2\sqrt{\varepsilon}}(\rho^{\mathfrak{E}}_{RA}|R) \leq E^{\varepsilon}(\mathfrak{E}) \leq H_0^{\varepsilon/2}(\rho^{\mathfrak{E}}_{RA}|R)$, for any ensemble $\mathfrak{E}$ and any $\varepsilon \geq 0$. This is done by standard tools like convexity arguments and the "gentle measurement" lemma [15].

As regards the asymptotic result of Theorem 2, the starting point is to note that the entanglement of formation itself can be expressed as a conditional entropy $E_F(\rho_{AB}) = \min_{\mathfrak{E}} H_r(\rho^{\mathfrak{E}}_{RA}|R)$, in close analogy with the expression (7) of the zero-error one-shot entanglement cost. Theorem 2 then reduces to the identity

$$\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\frac{1}{n}\min_{\mathfrak{E}_n} H_0^{\varepsilon}(\rho^{\mathfrak{E}_n}_{R_nA_n}|R_n) = \lim_{n\to\infty}\frac{1}{n}\min_{\mathfrak{E}_n} H_r(\rho^{\mathfrak{E}_n}_{R_nA_n}|R_n) \equiv E_F^{\infty}(\rho_{AB}), \tag{11}$$

where $\mathfrak{E}_n$ denotes a pure-state ensemble decomposition $\{p^n_i, |\phi^i_{A_nB_n}\rangle\}$ of $\rho^{\otimes n}_{AB}$, such that $\rho^{\otimes n}_{AB} = \sum_i p^n_i \phi^i_{A_nB_n}$, and $\rho^{\mathfrak{E}_n}_{R_nA_n} = \mathrm{Tr}_{\mathcal{H}_B^{\otimes n}}[\rho^{\mathfrak{E}_n}_{R_nA_nB_n}]$, with $\rho^{\mathfrak{E}_n}_{R_nA_nB_n}$ denoting the c-q extension of $\rho^{\otimes n}_{AB}$ as in Eq. (3). The identity (11) is proved by employing the information spectrum method [16], results of Ref. [17], and a generalized version of Stein's lemma established in Ref. [18].

F. B. acknowledges support from Japan's MEXT-SCF Program for Improvement of Research Environment for Young Researchers. N. D. acknowledges support from the European Community's Seventh Framework Program (FP7/2007-2013) under Grant Agreement No. 213681. This work was completed when F. B. was visiting the Statistical Laboratory of the University of Cambridge.

*buscemi@iar.nagoya-u.ac.jp
†n.datta@statslab.cam.ac.uk

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[2] F. Buscemi and N. Datta, J. Math. Phys. (N.Y.) **51**, 102201 (2010).
[3] See supplemental material at http://link.aps.org/supplemental/10.1103/PhysRevLett.106.130503 for the technical details.
[4] See, e.g., M. Ohya and D. Petz, *Quantum Entropy and Its Use* (Springer, New York, 1993), and references therein.
[5] R. Renner Ph.D. thesis, ETH Zurich, 2005.
[6] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[7] P. M. Hayden, M. Horodecki, and B. M. Terhal, J. Phys. A **34**, 6891 (2001).
[8] B. M. Terhal and P. Horodecki, Phys. Rev. A **61**, 040301 (2000).
[9] F. Buscemi and N. Datta, IEEE Trans. Inf. Theory **56**, 1447 (2010).
[10] If Λ is an LOCC transformation, $E_C^{(1)}(\rho_{AB};\varepsilon) \geq E_C^{(1)}[\Lambda(\rho_{AB});\varepsilon]$. To see this, note that, by Definition 1, there exists an LOCC map T that transforms $E_C^{(1)}(\rho_{AB};\varepsilon)$ singlets into a state $\tilde{\rho}_{AB}$ which is ε-close to ρ_{AB}. Then, by the monotonicity of the fidelity, $\Lambda(\tilde{\rho}_{AB})$ is also ε-close to $\Lambda(\rho_{AB})$, meaning that the composed LOCC map $\Lambda\circ T$ transforms $E_C^{(1)}(\rho_{AB};\varepsilon)$ singlets into a state which is ε-close to $\Lambda(\rho_{AB})$. This implies that $E_C^{(1)}[\Lambda(\rho_{AB});\varepsilon] \leq E_C^{(1)}(\rho_{AB};\varepsilon)$.
[11] C.-E. Mora, M. Piani, and H.-J. Briegel, New J. Phys. **10**, 083027 (2008).
[12] The presence of different smoothing parameters in the bounds of Theorem 1 arises from having defined the dilution error in terms of the fidelity and the *c*-*q* smoothing in terms of the trace norm. Such definitions have been adopted anyway, for the sake of overall simplicity.
[13] M. Hayashi, *Quantum Information: An Introduction* (Springer-Verlag, Berlin, 2006).
[14] G. Bowen and N. Datta, arXiv:0704.1957v1.
[15] A. Winter, IEEE Trans. Inf. Theory **45**, 2481 (1999); T. Ogawa and H. Nagaoka, in *Proceedings of the International Symposium on Information Theory, ISIT 2002* (IEEE, New York, 2002), p. 73.
[16] S. Verdu and T. S. Han, IEEE Trans. Inf. Theory **40**, 1147 (1994); T. Ogawa and H. Nagaoka, IEEE Trans. Inf. Theory **46**, 2428 (2000); M. Hayashi and H. Nagaoka, IEEE Trans. Inf. Theory **49**, 1753 (2003); G. Bowen and N. Datta, in *Proceedings of the International Symposium on Information Theory, ISIT 2006* (IEEE, New York, 2006), p. 45; H. Nagaoka and M. Hayashi, IEEE Trans. Inf. Theory **53**, 534 (2007).
[17] N. Datta, IEEE Trans. Inf. Theory **55**, 2816 (2009).
[18] F. G. S. L. Brandao and M. B. Plenio, Commun. Math. Phys. **295**, 791 (2010).