Universal Area Product Formulas for Rotating and Charged Black Holes in Four and Higher Dimensions

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We present explicit results for the product of all horizon areas for general rotating multicharge black holes, both in asymptotically flat and asymptotically anti–de Sitter spacetimes in four and higher dimensions. The expressions are universal, and depend only on the quantized charges, quantized angular momenta and the cosmological constant. If the latter is also quantized these universal results may provide a "looking glass" for probing the microscopics of general black holes.

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Explaining the origin of the black-hole entropy $S = \frac{1}{4}A$ at the microscopic level, where A is the area of the outer event horizon, is an outstanding problem for quantum theories of gravity. Significant insights have been achieved for supersymmetric, asymptotically flat, multicharged black holes in four and five dimensions [1], where the microscopic degrees of freedom can be explained in terms of a two-dimensional conformal field theory. More recent work has focused on the microscopic entropy of extreme rotating solutions [2]. By contrast, the detailed microscopic origin of the entropy of *nonextremal* rotating charged black holes remains an open problem, although recently there has been some promising progress [3].

Greybody factors (i.e., absorption coefficients) and radiation spectra provide another approach to probing the black-hole structure. An intriguing property of multicharged rotating black holes (in maximally supersymmetric supergravity theories) is that their wave equations are separable. The radial equation has poles at the locations of the horizons, where the radial component of the metric degenerates, with residues proportional to the inverse squares of the surface gravities, and so the Green functions are sensitive to the geometry near *all* the black-hole horizons, and not just the outermost one. The thermodynamic properties, including the surface gravity and area at each horizon, can therefore be expected to play a role in understanding the entropy at the microscopic level.

Some of these ideas have been explored for asymptotically flat, rotating, multicharged black holes in four and five spacetime dimensions. [Explicit solutions were given in [4,5], as generating solutions of maximally supersymmetric $\mathcal{N} = 4$ (or $\mathcal{N} = 8$) supergravities, obtained as toroidal compactifications of the heterotic string (or of Type IIA string or M theory).] In addition to their mass M, in four dimensions these solutions are specified by four charges Q_i ($i = 1, \cdots, 4$) and one angular momentum J, and in five dimensions by three charges Q_i ($i = 1, 2, 3$) and two angular momenta $J_{1,2}$. These black holes have just two horizons, and the area of the outer horizon has the tantalizing form [4]

$$S_+ = 2\pi(\sqrt{N_L} + \sqrt{N_R}), \tag{1}$$

where the integers N_L and N_R may be viewed as the excitation numbers of the left and right moving modes of a weakly-coupled two-dimensional conformal field theory. N_L and N_R depend explicitly on all the black-hole parameters. It was pointed out, first in the static case [6] and later for the general rotating black holes [7,8], that the entropy of the inner horizon, $S_- = \frac{1}{4}A_-$, is

$$S_- = 2\pi(\sqrt{N_L} - \sqrt{N_R}). \tag{2}$$

From this and (1), it follows that the product of the inner and outer horizon entropies satisfies $S_+S_- = 4\pi^2(N_L - N_R)$, which in terms of the underlying conformal field theory would be interpreted in terms of a level-matching condition. S_+S_- should therefore also be an integer [6–8]. (This point was recently reemphasized in [9].) It was found that S_+S_- is indeed quantized, and intriguingly, it is expressed solely in terms of the quantized charges and quantized angular momenta. In particular, it is modulus independent, taking the forms

$$S_+S_- = 4\pi^2\Big(\prod_{i=1}^{4} Q_i + J^2\Big) \tag{3}$$

$$S_+S_- = 4\pi^2\Big(\prod_{i=1}^{3} Q_i + J_R^2 - J_L^2\Big) = 4\pi^2\Big(\prod_{i=1}^{3} Q_i + J_aJ_b\Big) \tag{4}$$

in four and five dimensions, respectively. (These results were implicit in [7,8], though not explicitly evaluated.) The solutions considered here can be viewed as "seed solutions'' from which the complete families can be generated. The expressions for S_+S_- would be expressed in terms of S-, T-, and U-duality invariants built from the charges in the general case.

In a parallel development, Ansorg and collaborators [10–16] studied general axisymmetric stationary solutions of Einstein-Maxwell theory in four dimensions, with sources external to the horizons. They obtained striking ''universal'' formulas expressing the areas $A_\pm$ of the outer and inner Killing horizons in terms of the total angular momentum J and total charge Q. In particular, for Kerr-Newman black holes, they found (in the normalization conventions we use in the remainder of this Letter)

$$A_+^2 \leq A_+A_- = (8\pi J)^2 + (4\pi Q^2)^2, \tag{5}$$

in agreement (after conversion to our conventions) with the result given above in the special case that the four charges are set equal. Note the inequality (5) may be interpreted as a general criterion for extremality, and has been used to prove a No-Go theorem for the possibility of force balance between two rotating black holes [17].

It is natural to enquire whether analogous properties hold for more general classes of black holes, and especially, for those where the radial metric function has more than two zeroes. Examples include charged or rotating black holes in four or five-dimensional gauged supergravity, and in more than five dimensions with or without gauging. The wave equations in these backgrounds will have dominant contributions associated with poles at each of these zeroes. One can therefore again expect that the thermodynamics associated with *each* pole will play a role in governing the properties of the black hole at the microscopic level. At event horizons or Cauchy horizons, the metric at fixed radius has signature $(0, +, +, \cdots, +)$; that is, it describes a null hypersurface. However, it may happen that the induced metric has signature $(0, -, +, +, \cdots, +)$; in other words that the hypersurface is timelike, and the area of this ''pseudohorizon'' [18] is pure imaginary. The metric radial function may also have zeroes for complex values of the radial variable, these occur in conjugate pairs. In what follows, we shall just refer to zeroes of the radial function as horizons, regardless of whether the areas are real, imaginary or complex.

If it is indeed the case that geometries near all the horizons are involved in governing the microscopic behavior of the black hole, one might expect that the formulas (3) and (4) should generalize, for the more general black-hole examples, to expressions involving the products of *all* the horizon entropies or areas. This would suggest the possibility of an explanation for the microscopic behavior of such black holes in terms of a field theory in more than two dimensions.

We shall present results for the products of horizon areas in examples that include certain rotating black-hole solutions in gauged supergravities in dimensions 4, 5, 6, and 7, and also Kerr–anti-de Sitter rotating black holes in arbitrary spacetime dimensions. For the sake of brevity, we shall not present the details of our calculations in all cases, and instead, we have selected one example, namely, the rotating black hole in five-dimensional minimal gauged supergravity, for which we present the calculation of the area-product formula in more detail.

The formulas that we obtain for the area products are universal, they depend only on quantized charges, quantized angular momenta and the cosmological, or gauge-coupling, constant. In the case that the latter is also quantized (such as arises in compactifications of string theory, as discussed, for example, in [19]), these results are indeed suggestive of some underlying microscopics. For example, one may speculate that asymptotically anti–de Sitter black holes in four and five dimensions, for which there are three horizons, may have a microscopic origin in three-dimensional Chern-Simons theory.

We shall use normalization conventions where the Lagrangian density for gravity and Maxwell field(s) is of the form

$$\mathcal{L} = \frac{1}{16\pi G}\Big(R - \sum_i \Phi_i(\phi) F^i_{\mu\nu}F^{i\mu\nu} + (D-1)(D-2)g^2\Big), \tag{6}$$

where the functions of scalar fields (if present) are such that $\Phi^i(\phi)$ tends to unity at infinity for the black-hole solutions. We define charge(s) and angular momenta by

$$Q_i = \frac{1}{4\pi}\int \Phi^i(\phi) * F^i, \qquad J_i = \frac{1}{16\pi}\int *dK^i, \tag{7}$$

where $K^i = K^i_\mu dx^\mu$ and $K^{i\mu}\partial_\mu = \partial/\partial\psi_i$, where ψ^i is the azimuthal coordinate, with period 2π, in the 2-plane associated with the angular momentum J_i.

Our results for the products of the horizon areas for rotating black holes in gauged supergravities in dimensions 4, 5, 6, and 7 are as follows: $D = 4$ ungauged 4-charge [4]:

$$A_+A_- = (8\pi J)^2 + 256\pi^2 \prod_{i=1}^4 Q_i,$$

$D = 4$ gauged pairwise equal charges [20]:

$$\prod_{\alpha=1}^4 A_\alpha = (4\pi)^2 g^{-4}(8\pi J)^2 + 4g^{-4}(4\pi Q_1)^2(4\pi Q_2)^2.$$

$D = 5$ ungauged 3-charge [5]:

$$A_+A_- = (8\pi J_a)(8\pi J_b) + 256\pi \prod_{i=1}^3 Q_i,$$

$D = 5$ minimal gauged [21]:

$$\prod_{\alpha=0}^2 A_\alpha = -2i\pi^2 g^{-3}(8\pi J_a)(8\pi J_b) - ig^{-3}\Big(\frac{8\pi Q}{\sqrt{3}}\Big)^3,$$

$D = 5$ gauged $Q_1 = Q_2 \neq Q_3$ [22]:

$$\prod_{\alpha=0}^{3} A_\alpha = -\frac{2i\pi^2}{g^3}(8\pi J_a)(8\pi J_b) - \frac{i}{g^3}(8\pi Q_1)^2(8\pi Q_3).$$

$D = 6$ gauged [23]:

$$\prod_{\alpha=1}^{6} A_\alpha = g^{-8}\left(\frac{8\pi^2}{3}\right)^2(8\pi J_a)^2(8\pi J_b)^2 + g^{-6}\left(\frac{8\pi Q}{3}\right)^6.$$

$D = 7$ gauged [24]:

$$\prod_{\alpha=1}^{4} A_\alpha = \pi^3 g^{-5}\prod_{i=1}^{3}(8\pi J_i) - g^{-4}(2\pi Q)^4.$$

Note that we have included the cases of the 4-charge $D = 4$, and the 3-charge $D = 5$, solutions in ungauged supergravities, which were already presented as entropy-product formulas in the introduction. This is done for the sake of uniformity, using the normalization conventions that we follow in the rest of the body of the Letter. The citation in each heading above refers to the paper where the black-hole solution was constructed.

To illustrate how these calculations may be performed, we shall present the example of the rotating black hole in five-dimensional minimal gauged supergravity. The horizons are located at the roots of the radial function

$$\Delta(r) = (1 + g^2r^2)(r^2 + a^2)(r^2 + b^2) + q^2 + 2abq - 2mr^2 \tag{8}$$

that appears in the metric found in [21]. This is a cubic polynomial in r^2, and so there are six roots in total, occurring in pairs for which r^2 takes the same value. We may view $x = r^2$ as the radial variable, and thus just consider 3 roots. We may write Δ as

$$\Delta(r) = g^2\prod_{\alpha=0}^{2}(r^2 - r_\alpha^2). \tag{9}$$

The horizon areas are

$$A_\alpha = \frac{2\pi^2[(r_\alpha^2 + a^2)(r_\alpha^2 + b^2) + abq]}{\Xi_a\Xi_b r_\alpha}. \tag{10}$$

Using (8) and $\Delta(r_\alpha) = 0$, we can write this as

$$A_\alpha = -\frac{2\pi^2(2m + abqg^2)}{\Xi_a\Xi_b(1 + g^2r_\alpha^2)r_\alpha}\left[\frac{q(q + ab)}{2m + abqg^2} - r_\alpha^2\right]. \tag{11}$$

Noting from (8) and (9) that we may write $\prod_\alpha(c^2 - r_\alpha^2)$ as $g^{-2}\Delta(c)$, for any c, it is then straightforward to evaluate the product of the A_α. With the angular momenta and the charge given in terms of the rotation parameters a and b, the mass parameter m, and the charge parameter q by [21]

$$J_a = \frac{\pi[2am + qb(1 + g^2a^2)]}{4\Xi_a^2\Xi_b},\quad J_b = \frac{\pi[2bm + qa(1 + g^2b^2)]}{4\Xi_b^2\Xi_a},\quad Q = \frac{\sqrt{3}\pi q}{4\Xi_a\Xi_b}, \tag{12}$$

where $\Xi_a = 1 - a^2g^2$ and $\Xi_b = 1 - b^2g^2$, a straightforward calculation then gives the result we listed above. The calculations for the other examples can be performed in a similar manner.

For the Kerr-AdS metrics in arbitrary dimensions [25,26], it is necessary to separate the cases of even dimensions, $D = 2N + 2$, and odd dimensions, $D = 2N + 1$. In each case there are $2N + 2$ horizons and N angular momenta J_i. When $D = 2N + 1$, the radial metric function is a function of r^2, and the product over all horizons is equivalently expressible as the square of the product over just $N + 1$ horizons corresponding to a single choice of square root for each r_α^2. Our results for the horizon area products in D-dimensional Kerr-AdS are

$$D = 2N + 2:\ \prod_{\alpha=1}^{2N+2} A_\alpha = g^{-4N}(\mathcal{A}_{D-2})^2\prod_{i=1}^{N}(8\pi J_i)^2,$$
$$D = 2N + 1:\ \prod_{\alpha=0}^{N} A_\alpha = g^{-2N+1}c_N\mathcal{A}_{D-2}\prod_{i}(8\pi J_i),$$

where $c_N = (-1)^{(N+1)/2}$, and $\mathcal{A}_{D-2} = 2\pi^{(D-1)/2}/\Gamma[(D-1)/2]$ is the volume of the unit $(D-2)$ sphere.

The results presented above for black holes in gauged supergravities, and for Kerr-AdS black holes in pure gravity with a cosmological constant, admit straightforward limits to the ungauged, or zero cosmological constant, case. The radial functions in the metrics have a universal feature, as can be seen in (8) for the example of five-dimensional gauged supergravity, that the degree of the polynomial in r is reduced by 2 when the gauge-coupling g is set to zero. In this limit, the locations of these two "lost horizons" approach $r = \pm ig^{-1}$, and the areas of the lost horizons in the cases of even and odd dimensional black holes are

$$D = 2N + 2:\ A_{\rm lost} = (-1)^N g^{-2N}\mathcal{A}_{D-2},\quad D = 2N + 1:\ A_{\rm lost} = \mp i(-1)^N g^{-2N+1}\mathcal{A}_{D-2}. \tag{13}$$

If these areas are factored out from our previous expressions for the horizon area products, and then g is sent to zero, we can obtain the analogous formulas for the corresponding ungauged supergravities, and for asymptotically flat rotatating black holes in arbitrary dimensions. For the black holes in four and five-dimensional supergravities, the limits yield expressions encompassed by those given above for the ungauged cases. For the black holes in gauged six and seven dimensional supergravities, it is interesting to note that the electric charge terms scale to zero in the ungauged limit. The resulting expressions are then just the $D = 6$ and $D = 7$ specializations of the limiting forms for asymptotically-flat black holes in arbitrary dimensions, which we find to be

$$\begin{aligned} D = 2N + 2: \quad & \prod_{\alpha=1}^{2N} A_\alpha = \prod_{i=1}^{N} (8\pi J_i)^2, \\ D = 2N + 1: \quad & \prod_{\alpha=1}^{N} A_\alpha = \prod_{i} (8\pi J_i). \end{aligned} \tag{14}$$

We have also worked out the area-product formulas for a general class of charged rotating black holes in $D > 5$ ungauged supergravities [27], and we find the same phenomenon as in the $D = 6$ and $D = 7$ ungauged limits described above. Namely, the area products are independent of the charges in $D > 5$, and are given simply by the expressions (14) for uncharged asymptotically flat rotating black holes.

In this Letter, we have obtained formulas for the products of the horizon areas in a wide variety of black-hole solutions, showing that they are independent of moduli and are expressed solely in terms of quantized charges, angular momenta and the gauge-coupling constant. These provide tantalizing hints of a possible explanation for the microscopic properties of the black holes in terms of field theories in more than two dimensions.

The results in [10–16], where it was shown for certain four-dimensional cases that the universality of the product formulas persisted in the presence of external fields, suggest that our expressions in more general dimensions may also be robust in the presence of external fields. This is the analogue for black holes of the central idea of Old Quantum Theory, associated with the names of Bohr, Wilson, and Sommerfeld, that it is *adiabatic invariants* that should take quantized values because classically their values do not change under slow perturbations. It may be relatively straightforward to study the effect of external fields in four and five dimensions, since the symmetries allow a reduction to a system of equations on a two-dimensional quotient space. We hope to return to this subject in the future.

We believe that the quantized area-product formulas that we have obtained in this Letter provide a strong indication that there is a universal near-horizon structure for general black holes. This suggests the possibility that the microscopic degrees of freedom may admit a dual field-theoretic interpretation that generalizes the two-dimensional conformal field theory duals to certain four- and five-dimensional black holes, where the inner and out horizon areas are expressed in terms of the left and right conformal weights of the field-theory excitations. Further studies of black-hole dynamics, such as black-hole scattering processes, should provide further insights into this possibility.

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