Position Measurements Obeying Momentum Conservation

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We present a hitherto unknown fundamental limitation to a basic measurement: that of the position of a quantum object when the total momentum of the object and apparatus is conserved. This result extends the famous Wigner-Araki-Yanase theorem, and shows that accurate position measurements are only practically feasible if there is a large momentum uncertainty in the apparatus.

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Introduction.—The extent to which the elements of the quantum mechanical formalism relate to physically measurable quantities has been the subject of many investigations in the history of quantum mechanics. It is well known, for example, that not all self-adjoint operators represent observables in the presence of superselection rules. Wigner [1] showed that a different type of measurement limitation arises due to conservation laws for quantities that are additive over the system plus apparatus. Specifically he, and subsequently Araki and Yanase [2] proved that a discrete self-adjoint operator not commuting with such a conserved quantity does not admit perfectly accurate and repeatable measurements. The original proofs of the WAY theorem are restricted to cases where the object part of the conserved quantity is bounded. If that quantity is assumed to be discrete, the second, positive part of the WAY theorem asserts that a repeatable measurement can be approximately realized, but this comes at a price: high accuracy requires a large size (suitably defined) of the apparatus [2,3].

The most comprehensive extensions of the WAY theorem obtained so far [4,5] do not encompass more general cases including continuous-spectrum and unbounded observables. In fact, it is a fundamental result established by Ozawa [6] that continuous observables do not admit any repeatable measurements, irrespective of whether there are additive conserved quantities.

Nevertheless, our analysis of a model presented by Ozawa [7] in this journal leads us to conclude that WAY-type limitations do exist for measurements of continuous quantities, contrary to the view expressed there. We show for the prototypical example of position measurements obeying momentum conservation that the accuracy and approximate repeatability of such measurements are limited by the finite size of the apparatus if it is assumed that the pointer observable commutes with the momentum. This condition, which following Ozawa [8] we call the Yanase condition, is certainly significant but often neglected: in order to secure reproducible measurement records, it is necessary that the pointer observable itself can be measured repeatably and accurately. Insofar as the WAY theorem applies to the pointer observable being measured, this may only be achieved if that observable commutes with the conserved quantity.

We also consider an alternative model which shows, perhaps surprisingly, that if one relinquishes the Yanase condition, position measurements obeying momentum conservation may be possible with arbitrary accuracy and good repeatability properties, without any constraint on the size of the apparatus. This stands in contrast to the discrete-bounded case where a measurement of a quantity not commuting with an additive conserved quantity can neither be repeatable nor satisfy the Yanase condition [9]. We also provide a general, model-independent argument corroborating these findings.

A thorough understanding of such quantum limitations to measurements is crucial; from a foundational perspective it provides a more complete description of physical reality as it manifests itself through observation, and from a pragmatic viewpoint it delineates the possible fundamental obstacles that must be accounted for in technological applications. Ozawa and co-workers [10] have demonstrated a limitation to the realizability of quantum logic gates insofar as the observables involved are subject to the WAY theorem. Similarly it must now be expected that operations for continuous-variable quantum information processing tasks are only realizable to a limited accuracy in the presence of an additive conservation law, given that there will typically be a need to limit the size of the component systems. For accurate position measurements subject to a WAY-type limitation, a large momentum spread—and thus kinetic energy—is required in the apparatus, which conflicts with the low temperatures necessary for the control of a quantum system.

In the models discussed below, the system and the apparatus are particles in one space dimension, represented by the Hilbert space of square-integrable functions on $\mathbb{R}$. We will work in units where $\hbar = 1$.

Ozawa's model.—In [7], Ozawa claimed that there is no WAY-type limitation to position measurements. He introduced a model involving four particles with position operators Q, $Q_{\mathcal{A}}$, $Q_{\mathcal{B}}$, $Q_{\mathcal{C}}$. The interaction Hamiltonian is translation invariant and thus conserves total momentum;

the resulting unitary time evolution operator for a time interval τ is

$$U = \exp\left[-i\frac{K}{2}\tau(Q - Q_{\mathcal{A}})(Q_{\mathcal{B}} - Q_{\mathcal{C}})\right]. \qquad (1)$$

U acts on $\mathcal{H}_{\mathrm{total}} := \mathcal{H} \otimes \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{C}}$ and we adopt the obvious shorthand (e.g., $Q = Q \otimes \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_{\mathcal{B}} \otimes \mathbf{1}_{\mathcal{C}}$) for simplicity of notation. The constant K describes the coupling strength; we will use the abbreviation $K\tau = \lambda$.

The aim is to use this interaction to measure a particle's position Q by transcribing the Q distribution to a pointer observable Z on an apparatus that is accessible to an experimenter. Here the pointer is taken to be the relative momentum $Z = P_{\mathcal{C}} - P_{\mathcal{B}}$. With this choice, particle $\mathcal{A}$ appears as an auxiliary "reference" system by which information about Q can be recovered. It is also clear that $[Z, P_{\mathrm{total}}] = 0$ where P_{total} is the sum of the momenta of the system and apparatus. Thus the Yanase condition is satisfied in this model.

Ozawa chooses $K\tau = 1$ (which makes position and momentum dimensionless) and the initial apparatus state $\xi = |Q_{\mathcal{A}} = \bar{y}\rangle \otimes |P_{\mathcal{C}} - P_{\mathcal{B}} = \bar{y}\rangle$ for $\bar{y}$ constant. He omits the state representing the final degree of freedom pertaining to $P_B + P_C$, which does not alter the outcome. By the uncertainty relation, this choice of (unnormalizable) initial state ξ cannot have finite momentum spread.

The observable-to-be-measured Q is preserved by the interaction: $Q = Q(\tau)$. The characteristic function, which arises as the Fourier transform of the joint probability density of $Q = Q(\tau)$ with the time-evolved pointer observable $Z(\tau) = (P_{\mathcal{C}} - P_{\mathcal{B}}) + (Q - Q_{\mathcal{A}})$, is given by the expression $\langle \varphi \otimes \xi | \exp(i(\mu Q(\tau) + \mu' Z(\tau)))\varphi \otimes \xi\rangle$. Ozawa gives this in integral form as

$$\iint e^{i(\mu x + \mu' z)} |\varphi(z)|^2 \delta(x - z) dx dz, \qquad (2)$$

where z denotes a spectral value of Z and φ is the preparation of the system. However, this follows only by ignoring the twofold infinity generated by the term $\langle\bar{y}|\bar{y}\rangle\langle\bar{y}|\bar{y}\rangle$ that would appear in the original expression for the characteristic function. Thus the distribution $|\varphi(z)|^2\delta(x - z)dxdz$ following from (2) is not the joint distribution of $Q(\tau)$ and $Z(\tau)$, and hence it does not follow that this model realizes an accurate and repeatable measurement of position. This conclusion is in line with Ozawa's result that continuous observables do not admit repeatable measurements [6].

We shall now calculate the relevant measurement probabilities directly in the Schrödinger picture, using normalizable states only [11]. It follows that the measurement accuracy—and degree of repeatability—are limited by the "size" of the apparatus, in close analogy to what we referred to as the positive part of the WAY theorem in the case of discrete quantities. Here we use the position and momentum representations for the initial (product) state, $\Psi_0(x, y, u, v) = \varphi(x)\Phi_1(y)\Phi_2(u)\phi(v)$ with u and v denoting spectral values of $P_{\mathcal{C}} - P_{\mathcal{B}}$ and $P_{\mathcal{B}} + P_{\mathcal{C}}$, respectively. After a time τ (which we will also write as λ/K), the state has evolved into

$$\Psi_\tau(x, y, u, v) = \varphi(x)\Phi_1(y)\Phi_2\left(u + \frac{1}{2}\lambda(x - y)\right)\phi(v). \qquad (3)$$

The probability density for u is obtained as a marginal from the joint density for the time-evolved state Ψ_τ;

$$p_{\Psi_\tau}(u) = \iiint |\Psi_\tau(x, y, u, v)|^2 dx dy dv. \qquad (4)$$

The probability for the pointer to assume a value in a set S is

$$\mathsf{P}_{\Psi_\tau}(u \in S) = \int_S du \int dx \int dy |\varphi(x)|^2 |\Phi_1(y)|^2 \times \left|\Phi_2\left(u + \frac{1}{2}\lambda(x - y)\right)\right|^2 \int dv |\phi(v)|^2. \qquad (5)$$

We introduce a *scaling function* $f:\mathbb{R} \to \mathbb{R}$ to allow for the measured observable and the pointer observable to have different scales. With $f(u) = -(2/\lambda)u$ and putting $S = f^{-1}(X) = -(\lambda/2)X$ (the set of all u with $f(u) \in X$), the right hand side of (5) can be written as:

$$\int dx |\varphi(x)|^2 \chi_X \star e^{(\lambda)}(x) = \int_X \int dx' |\varphi(x + x')|^2 e^{(\lambda)}(x') \equiv \mathsf{P}_\varphi(x \in X), \qquad (6)$$

with $\star$ denoting the convolution and χ_X the set indicator function. The function $e^{(\lambda)}$ is a density and takes the form $e^{(\lambda)}(x) = (|\Phi_1|^2 \star |\Phi_2^{(\lambda)}|^2)(x)$, where $\Phi_2^{(\lambda)}(s) = \sqrt{\lambda}\Phi_2(\lambda s)$. This density function $e^{(\lambda)}$ represents the inaccuracy of the measurement [12], in the sense that the actual probability density appearing in (6) is a smearing of the ideal position probability density $|\varphi(x)|^2$; we see that the narrower the width of $e^{(\lambda)}$, the more accurate the measurement. In the extreme case that $e^{(\lambda)}$ tends to a delta function, the probabilities (6) become those of an accurate position measurement.

We compute $\mathrm{Var}(e^{(\lambda)}) = \mathrm{Var}|\Phi_1|^2 + \frac{4}{\lambda^2}\mathrm{Var}|\Phi_2|^2$. Thus the variance of $e^{(\lambda)}$ does not vanish in the limit $\lambda \to \infty$ but is given by the variance of the $Q_{\mathcal{A}}$ distribution in the "reference system" state Φ_1; by virtue of the uncertainty relation for $Q_{\mathcal{A}}$ and $P_{\mathcal{A}}$, this can only be made small at the expense of making the width of the $P_{\mathcal{A}}$ distribution large. We see that in order to recover accurate information about the particle's position Q, it is the reference position $Q_{\mathcal{A}}$ that needs to be highly localized, independently of the momentum spread of the pointer.

In accordance with the findings of Yanase [3] for the case where the object part of the conserved quantity was bounded and discrete, we see here that the size of the apparatus limits the position measurement accuracy.

A more useful measure of inaccuracy than the variance of a distribution e is given by the overall width $W(e; 1-\varepsilon)$ of e at confidence level $1-\varepsilon$, defined as the smallest possible size of a suitably located interval J such that the probability $\int_J e(q)dq \geq 1-\varepsilon$. In contrast to the variance, the overall width is finite whenever $\varepsilon > 0$.

It is straightforward to show that the overall width of a convolution of two probability distributions is bounded below by the width of the largest. In the case of the Ozawa model, we thus see that the overall width of $e^{(\lambda)}$ is always bounded below by the overall width of the distribution $|\Phi_1|^2$, which is independent of λ. This generalizes the above argument which used variances.

An alternative model.—Next we revisit a position measurement model ([13], Sec. IV.3.3) that violates the Yanase condition. Momentum conservation is implemented via the unitary coupling

$$U = \exp\left[-i\frac{\lambda}{2}((Q-Q_{\mathcal{A}})P_{\mathcal{A}} + P_{\mathcal{A}}(Q-Q_{\mathcal{A}}))\right], \quad (7)$$

which acts on $\mathcal{H} \otimes \mathcal{H}_{\mathcal{A}}$. As before, λ is a shorthand for $K\tau$ where K is the coupling strength and τ the duration of the interaction. Here λ is naturally dimensionless. The pointer observable is $Q_{\mathcal{A}}$, which of course does not commute with the total momentum.

We can again extract the probability density for the pointer after time τ, with $\Psi_\tau = U(\varphi \otimes \phi)$:

$$p_{\Psi_\tau}(y) = \int |\Psi_\tau(x,y)|^2 dx. \quad (8)$$

The form of the final state $\Psi_\tau(x,y)$ gives the pointer probabilities

$$\mathsf{P}_{\Psi_\tau}(y \in f^{-1}(X)) = \int_{f^{-1}(X)} dy \int dx |\varphi(x)|^2 \times e^{\lambda}|\phi(ye^{\lambda} - x(e^{\lambda}-1))|^2, \quad (9)$$

which, with $f^{-1}(X) := (1-e^{-\lambda})X$, we write in the form

$$\int dx |\varphi(x)|^2 \chi_X \star e^{(\lambda)}(x) \equiv \mathsf{P}_\varphi(x \in X). \quad (10)$$

The probability density $e = e^{(\lambda)}$ now takes the form $e^{(\lambda)}(x) = (e^{\lambda}-1)|\phi(-x(e^{\lambda}-1))|^2$. The scaling behavior is thus exponential in λ; the inaccuracy width scales with $e^{-\lambda}$ and an arbitrarily accurate measurement of Q is feasible without any constraint on the size of the apparatus.

Repeatability.—It is worth elucidating further the differences between the two models studied here. The first, which satisfied the Yanase condition, displayed limitations to the accuracy of a position measurement that could be overcome only by allowing the reference system to have large momentum. The second, which manifestly violated the Yanase condition, imposed no such constraint and arbitrary accuracy could be achieved by a tuning of the interaction strength. However, as in the original work [1,2], it is not only the measurement accuracy that plays a prominent role, but also the repeatability properties, which we discuss now.

We shall confine the probe's initial state wave functions to a bounded subset of the real line. This is not an overly stringent requirement from a physical perspective. In the Ozawa model this simply amounts to the initial state functions $\Phi_1(y)$ and $\Phi_2(u)$ having finite extent in the relevant variables; in the second model it means that the probe state function $\phi(y)$ is concentrated in a finite interval. Thus we can think of the density $e^{(\lambda)}$ as being concentrated on the interval $[-d, d]$ in either model.

One way of quantifying the degree of approximate repeatability [14,15] in the case of a position measurement is as follows: a measurement is said to be approximately, or δ repeatable if given an outcome in a set X, the outcome of an immediate subsequent control measurement will be found, with probability 1, in a suitably enlarged set X_δ (where X_δ is the set of points not more than a distance $\delta > 0$ away from X). This can be written symbolically as a conditional probability of finding the particle's position $x \in X_\delta$ given that the pointer was found to have a value $u \in f^{-1}(X)$:

$$\mathsf{P}_{\Psi_\tau}(x \in X_\delta | u \in f^{-1}(X)) = 1 \quad (11)$$

for all sets X. Considering the control measurement to be accurate, for this to be satisfied in the Ozawa model we must have $\chi_X \star e^{(\lambda)}(x) = 0$ whenever x is outside X_δ, and this follows if $\delta \geq d$. If the initial apparatus states Φ_1 and Φ_2 are concentrated on intervals $[-\ell, \ell]$ and $[-m, m]$, respectively, we have that $d = \ell + m/\lambda$. Therefore even as the coupling strength λ becomes large, δ is bounded below by the width of the reference system state Φ_1, and in order to recover good repeatability properties (i.e., a small δ), the state Φ_1 must carry a large spread of momentum.

In the alternative model we see similar behavior, with a fundamental difference; we again have that $\delta \geq d$ enables approximate repeatability in the sense of (11). However, in contrast to the Ozawa model, simply letting λ be large allows for arbitrarily good repeatability; if ϕ is concentrated on $[-n, n]$, then $d = n/(e^{\lambda}-1)$.

Thus under violation of the Yanase condition, arbitrarily accurate and repeatable information transfer from the system to a quantum probe is feasible without any size constraint (n can be arbitrarily large, allowing the spread of the probe momentum to be small).

General argument.—Finally we adapt an approach due to Ozawa [8] to obtain a generic, model-independent trade-off between the qualities of accuracy and repeatability on one hand and the necessary size of the apparatus on the other. The noise operator N is defined as $N := Z(\tau) - Q$, where $Z(\tau)$ represents the Heisenberg-evolved pointer observable after the interaction period τ. One then defines the noise $\epsilon(\varphi)^2 := \langle \varphi \otimes \phi | N^2 \varphi \otimes \phi \rangle \equiv \langle N^2 \rangle$. Clearly $\epsilon(\varphi)^2 \geq (\Delta N)^2$. For a measurement scheme to represent an approximation to a position measurement, it is reasonable to require that the noise is finite across all input object states. Thus the supremum $\epsilon := \sup \epsilon(\varphi)$ should be finite and would then give a global measure of error. The uncertainty relation then gives

$$\epsilon^2 \geq \epsilon(\varphi)^2 \geq \frac{1}{4} \frac{|\langle [Z(\tau) - Q, P + P_{\mathcal{A}}] \rangle|^2}{(\Delta P_{\text{total}})^2}, \qquad (12)$$

where $(\Delta P_{\text{total}})^2 = (\Delta_\varphi P)^2 + (\Delta_\phi P_{\mathcal{A}})^2$. This inequality entails a measurement limitation whenever the right-hand side is nonzero for some object states. It is also evident that if the numerator is nonzero, the only way of making this lower bound to the error small independently of the object properties is by making the momentum variance $(\Delta_\phi P_{\mathcal{A}})^2$ of the apparatus large.

The vanishing of the numerator for all object states φ follows when the commutator is zero, which happens just when the pointer at time 0 satisfies $[Z, P_{\mathcal{A}}] = i$. This is the case in the second model discussed above where a WAY-type limitation was found to be absent.

If the Yanase condition is stipulated, one obtains $[Z(\tau) - Q, P + P_{\mathcal{A}}] = i$, and (12) yields

$$\epsilon^2 \geq [2\Delta_\phi P_{\mathcal{A}}]^{-2}. \qquad (13)$$

This bound only allows for an increase in accuracy when $(\Delta_\phi P_{\mathcal{A}})^2$ is large, thus establishing the necessity of the large apparatus size for good measurements.

An attempt at capturing (approximate) repeatability in the generic case follows from considering the quantity $\mu(\varphi)^2 := \langle \varphi \otimes \phi | (Q(\tau) - Z(\tau))^2 \varphi \otimes \phi \rangle$; intuitively if this expectation is small, then the difference between the measured observable and the time-evolved system observable is small, and hence the measurement should display some level of repeatability. An argument analogous to that above gives, for $\mu^2 := \sup \mu(\varphi)^2$

$$\mu^2 \geq [2\Delta_\phi P_{\mathcal{A}}]^{-2}. \qquad (14)$$

This provides an indication that under the Yanase condition, good repeatability is achieved, again, only when there is a large momentum variance in the apparatus. It remains to be shown that these conclusions persist when more operationally significant measures of inaccuracy and repeatability are used, such as those in [16]. For example, a new measure of repeatability may be formulated via the repeatability width, defined as the smallest δ such that a repeatability condition like (11) is satisfied, possibly only up to probabilities no less than a threshold $1 - \varepsilon$.

In conclusion, evidence for a WAY-type theorem for continuous unbounded quantities has been provided through two models of momentum-conserving position measurements and two model-independent inequalities. The analysis entails also that no such limitation arises if only relative distances are measured, that is the distance between the object and the "reference system," which is provided by the measuring apparatus. When this is incorporated into the quantum description, the conservation law can be manifestly satisfied for the combined object-apparatus system, with the measured observable as the relative position. In this case, the Yanase condition must be satisfied for good accuracy to be achieved. This points to a possible connection, hinted at by Aharonov and Rohrlich [17], with the theory of superselection rules and quantum frames of reference, a subject of renewed interest in the past decade [18], which seems to deserve further systematic study.

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