Quantized Pumping and Topology of the Phase Diagram for a System of Interacting Bosons

Erez Berg,[1] Michael Levin,[1,2] and Ehud Altman[3]

[1]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
[2]*Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA*
[3]*Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot, 76100, Israel*

(Received 9 August 2010; revised manuscript received 6 February 2011; published 17 March 2011)

Interacting lattice bosons at integer filling can support two distinct insulating phases, which are separated by a critical point: the Mott insulator and the Haldane insulator [E. G. Dalla Torre, E. Berg, and E. Altman, Phys. Rev. Lett. **97**, 260401 (2006).]. The critical point can be gapped out by breaking lattice inversion symmetry. Here, we show that encircling this critical point adiabatically pumps one boson across the system. When multiple chains are coupled, the two insulating phases are no longer sharply distinct, but the pumping property survives. This leads to strict constraints on the topology of the phase diagram of systems of quasi-one-dimensional interacting bosons.

DOI: 10.1103/PhysRevLett.106.110405 PACS numbers: 05.30.Jp, 03.75.Kk, 03.75.Lm, 21.60.Fw

In the early 1980s, Thouless [1] made the surprising observation that certain band insulators can sustain dissipationless and quantized charge transport by adiabatic pumping. The classic example of this effect is seen in a half-filled tight binding chain with two sites per unit cell [1]. As parameters of the Hamiltonian are changed adiabatically along a closed loop around the single gapless point in the two parameter space, a unit charge is transported through the chain. This simple observation had interesting implications to other systems. For example, it was quickly realized [2,3], that Laughlin's original argument for the quantization of the Hall conductance may be formulated in the same mathematical terms as the pumping problem. In connection with more recent developments, the ideas of topological pumping through band insulators were precursors of the theoretical [4–7] and subsequent experimental [8,9] discovery of topological band insulators. Indeed, the Z_2 topological invariant associated with these systems can be reformulated in terms of adiabatic pumping [10].

Although quantized pumping has been discussed primarily in the context of noninteracting fermions, the concept is much more general. The pumped charge can be formulated in terms of a topological Chern number associated with parallel transport of the many-body wave function in Hilbert space [2,3]. In particular this formulation ensures robustness of the quantization to disorder and interaction and also enables direct extension of the concepts to spin pumping in spin-1/2 chains [11]. All these extensions are adiabatically connected to the case of a band insulator, either directly or via a Jordan-Wigner transformation.

In this Letter we show that a natural model of interacting lattice bosons at integer filling, which is not directly mappable to a band insulator, allows quantized transport through Mott insulating states by adiabatic pumping. The existence of nontrivial loops in the gapped regions of parameter space defines a topological index, which may be associated with the gapless (superfluid) phases they surround. It also sets constraints on the structure of the phase diagram, or more precisely, on the topology of the gapless regions within it.

The basic model we consider is an extended Bose Hubbard model (EBHM), at integer filling, on coupled chains

$$H = \sum_{\alpha}[H_{\alpha} + H_{\lambda,\alpha} + H_{\perp,\alpha}], \tag{1}$$

where

$$H_{\alpha} = \sum_{j}\Big[-t(b^{\dagger}_{\alpha,j}b_{\alpha,j+1} + \text{H.c.}) + \frac{U}{2}n_{\alpha,j}(n_{\alpha,j}+1)\Big] + V\sum_{j}n_{\alpha,j}n_{\alpha,j+1}, \tag{2}$$

is a single chain Hamiltonian defined on chain α. $b^{\dagger}_{\alpha,j}$ creates a boson at position j in chain α, and $n_{\alpha,j} \equiv b^{\dagger}_{\alpha,j}b_{\alpha,j}$. The Hamiltonian

$$H_{\lambda,\alpha} = \lambda\sum_{j}[n_{\alpha,j}b^{\dagger}_{\alpha,j}b_{\alpha,j+1} - n_{\alpha,j+1}b^{\dagger}_{\alpha,j+1}b_{\alpha,j} + \text{H.c.}], \tag{3}$$

is a perturbation that breaks the bond-centered inversion symmetry of H_{α}. Finally,

$$H_{\perp,\alpha} = \sum_{j}[-t_{\perp}(b^{\dagger}_{\alpha,j}b_{\alpha+1,j} + \text{H.c.}) + V_{\perp}n_{\alpha,j}n_{\alpha+1,j}] \tag{4}$$

denotes interchain coupling. The model (1) or related Hamiltonians can be realized with ultracold dipolar molecules or atoms with optically induced dipole moments [12]. Crucial for our analysis is the presence of the perturbation λ, which breaks the inversion symmetry of the chain. It will be naturally generated if the underlying optical potential is not symmetric under inversion. Such a lattice potential can be produced by two lasers, one of which has double the wavelength of the other. In one extreme limit this configuration gives rise to a lattice of double well potentials [13,14], which indeed are not inversion symmetric in general.

A single chain.—We have shown previously, that the EBHM on a single chain [Eq. (2)] exhibits a quantum phase transition from a Mott insulating (MI) state to a novel gapped phase, which we termed a "Haldane insulator" (HI), upon increasing the nearest neighbor interaction [15]. Both phases are completely disordered in the sense that they do not break any symmetry of the Hamiltonian. The new state is analogous to the Haldane gapped state of spin-1 chains, and is characterized by a string order parameter, albeit in the boson density rather than the spin. It was later shown [16–19], that the distinction between the HI and MI phases is protected by lattice inversion symmetry. A perturbation, such as $H_{\lambda,\alpha}$ above, which breaks the inversion symmetry about a bond, opens a gap at the HI-MI transition and allows adiabatic connection between the two gapped phases. Thus, in the two parameter space (V, λ) the transition becomes an isolated critical point. We shall argue that an adiabatic passage around the critical point entails transport of a single boson through the chain.

To see this we turn to the long-wavelength description of the extended Hubbard chain, with the inversion symmetry breaking perturbation λ. Near to the HI-MI phase transition it is given by the following sine-Gordon field theory [16]

$$H_0 = \frac{u}{2\pi}\int dx\Big[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\phi)^2 - g\cos(2\phi) - \lambda\sin(2\phi)\Big], \tag{5}$$

with the Luttinger parameter K in the regime $1/2 < K < 2$. The parameter g is in general a complicated function of the microscopic interactions. $g(U, V, t) > 0$ in the MI phase, $g < 0$ in the HI and vanishes on the critical line separating these two phases. A naive continuum limit gives the approximate dependence $g \approx U/2 - V$ [16]. Here $\partial_x\phi/\pi = \rho$ is the long-wavelength component of the boson density, θ is its dual field satisfying $[\partial_x\phi(x), \theta(x')] = i\pi\delta(x - x')$, and u is the sound velocity. Note that under inversion, $\rho(x) \to \rho(-x)$; therefore, $\phi(x) \to -\phi(-x)$, which makes it clear that the λ term is odd under inversion. The last two terms can be written compactly as $\tilde{g}\cos(2\phi - \chi)$, where $\tilde{g} = \sqrt{g^2 + \lambda^2}$ and $\chi = \arctan(\lambda/g)$. In the regime of interest $K < 2$, making the cosine term relevant. $\cos(2n\phi)$ and $\sin(2n\phi)$ with $n > 1$ may also appear in H_0, but we assume that these terms are irrelevant at the HI-MI critical point, $(\lambda = 0, g = 0)$.

The critical point is entirely surrounded by a gapped state [see Fig. 1(a)] in which the field ϕ is essentially locked to the value $\chi/2$. Therefore an adiabatic change of the system parameters, which takes it in a counterclockwise loop around the critical point, incurs a continuous change of $\phi(x)$ by π everywhere in space. By definition of the field, $\phi(x)$ suffers a π shift every time a particle passes through x. The last observation implies the transport of exactly one boson from left to right in a counterclockwise loop. Another way to derive the quantization is to refermionize the field theory (5). The quantized charge can be computed directly for $K = 1$, which maps to free fermions [20]. It follows for other values of K by adiabatic continuity.

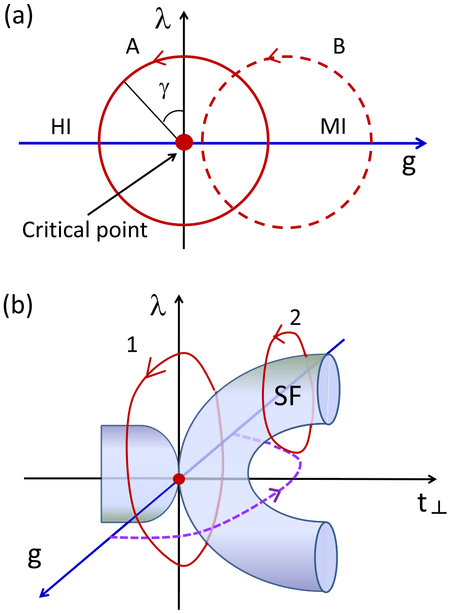

FIG. 1 (color online). Phase diagram topology. (a) Phase diagram of a single chain. The parameter g tunes across the Haldane (HI) to Mott (MI) insulator transition. These two phases are sharply defined only in the presence of inversion symmetry ($\lambda = 0$). A closed adiabatic path which encircles the critical point (path A) entails pumping of a single boson across the insulator. (b) Schematic phase diagram of two coupled chains. The MI and HI phases can be adiabatically connected via the dashed path. However, since path 1 pumps one boson per chain, it cannot be collapsed adiabatically to a point without crossing the gapless region. Path 2 entails the pumping of one boson per two chains.

To enable continuous pumping, the chain must be connected to gapless reservoirs. This arises naturally in a realization using an optical lattice and a harmonic trap in which the incompressible phase will be flanked by superfluid wings. The adiabaticity condition needed to ensure quantized pumping is $\dot{\chi} \ll \Delta \sim \Lambda(\tilde{g}/\Lambda)^{1/(2-K)}$ [20], where Δ is the gap along the cycle. Λ, the ultraviolet cutoff of the continuum theory, is of the order of the bandwidth $2t$.

The topological character of the pumped charge makes it robust to small perturbations of the Hamiltonian [2]. In particular, for the case of many weakly coupled chains, driving all chains adiabatically along loop A still pumps one boson per chain. For arbitrary coupling between chains, we shall see that the quantization of the pumped charge imposes stringent constraints on the topology of the phase diagram in the enlarged parameter space. We demonstrate this below using the example of two coupled chains and then comment on generalizations to any number of coupled chains.

Two coupled chains.—The critical point at $(g, \lambda) = 0$ is unstable to weak tunnel coupling $t_\perp$ between a pair of chains [16]. Using a renormalization group analysis we have shown that the critical point expands to a gapless phase (Luttinger liquid) with radius $\sim t_\perp^\eta$ around the origin in the space (g, λ), where the precise exponent η is given in Ref. [16].

How is the pumped charge associated with an adiabatic cycle around the critical point, affected by turning on the interchain coupling $t_\perp$? As long as the path encircles the gapless region from the outside, then it is adiabatically connected to the nontrivial pumping cycle around the HI-MI critical point of the decoupled chains. The topological Chern number cannot change and hence the pumped charge must remain quantized at one boson per chain upon encircling the gapless region. Below we address the evolution of the gapless region for increasing interchain coupling beyond weak coupling.

To understand how the phase diagram evolves with stronger values of interchain coupling we should take into account another crucial fact. For two chains there is no sharp distinction between the HI and MI phases, even in the presence of inversion symmetry ($\lambda = 0$) [18]. This means that the HI and MI phases of two decoupled chains can be connected adiabatically by a path in Hamiltonian space going through a region with nonzero interchain coupling. We demonstrate this explicitly using a density matrix renormalization group (DMRG) calculation of the following spin-1 ladder model:

$$\begin{aligned} H_{\text{spin}} = \sum_{i,\alpha} [&V S^z_{\alpha,i} S^z_{\alpha,i+1} - t(S^+_{\alpha,i} S^-_{\alpha,i+1} + \text{H.c.}) + U(S^z_{\alpha,i})^2 \\ &+ \lambda(S^z_{\alpha,i} S^+_{\alpha,i} S^-_{\alpha,i+1} - S^z_{\alpha,i+1} S^+_{\alpha,i+1} S^-_{\alpha,i} + \text{H.c.})] \\ &+ \sum_i [V_\perp S^z_{1,i} S^z_{2,i} - t_\perp (S^+_{1,i} S^-_{2,i} + \text{H.c.})]. \end{aligned} \tag{6}$$

This model can be thought of as a truncation of the EBHM (1) to the space of the three lowest occupation states $n_i = S^z_i + 1$ [21]. Crucially, the two models have the same low energy limit [16,22].

Figure 2 shows the phase diagram of the model (6), as a function of $t_\perp$, λ, and U, which is used to tune the MI-HI transition. (U is related to g in Eq. (5) by $g \propto U - U_c$, where U_c is the location of the MI-HI transition.) We have fixed $V = 2t$ and $V_\perp = 2t_\perp$. The phase diagram was determined by measuring the spin gap, $\Delta_s = E(S = 1) - E(S = 0)$, and extrapolating it to the thermodynamic limit. System sizes of up to $L = 64 \times 2$ were used, keeping $m = 200$ states.

We see in Fig. 2, that upon increasing the interchain coupling $t_\perp$ and $V_\perp$ the HI-MI critical point first expands to a gapless region as predicted by the weak coupling theory [16], but collapses at stronger coupling, allowing for an adiabatic connection between the HI and MI states. The fact that the gapless region ends may at first seem contradictory to our previous assertion that an adiabatic loop around this region in the space (g, λ) entails pumping of one boson per chain. If the gapless region ends, and the loop can be collapsed adiabatically to a trivial point in the gapped state with increasing interchain coupling, how can it sustain a nontrivial Chern number?

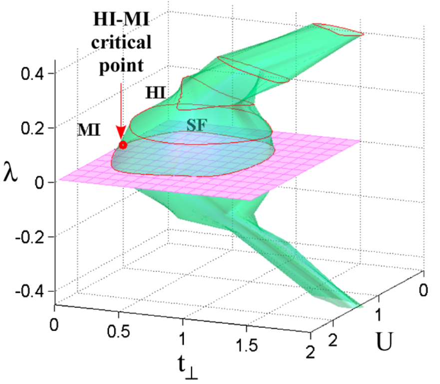

FIG. 2 (color online). Phase diagram of the spin-1 two-leg ladder defined in Eq. (6) as a function of U, $t_\perp$, and λ, calculated using DMRG. We have fixed $V_\perp = 2t_\perp$. For $t_\perp = \lambda = 0$, there are two distinct gapped phases, the HI and MI, which are separated by a critical point at $U \approx 1$. Upon turning on $t_\perp$, the critical point expands to a finite superfluid (SF) region, and the two gapped phases are not sharply distinct. For $\lambda > 0$, the MI-HI critical point at $t_\perp = 0$ becomes gapped. The gapless region shrinks upon increasing λ, but does not disappear. The phase diagram has the same topology as in Fig. 1(b).

To avoid this contradiction, the gapless region must split into two branches in the $\pm\lambda$ directions, which either extend indefinitely, or terminate discontinuously on a 1st order transition plane. With this topology, a loop surrounding the original critical point at $t_\perp = 0$ cannot be collapsed adiabatically into a point. The numerically obtained phase diagram in Fig. 2 is consistent with these considerations: although the superfluid region in the $\lambda = 0$ plane is finite, it has two branches which extend in the $\pm\lambda$ directions. These branches do not terminate up to the largest values of λ we examined (in [20] we present results for higher λ values).

Given the topology of the gapless phase it is natural to ask what is the pumped charge associated with a path surrounding only one of the two branches at either positive or negative λ [path 2 in Fig. 1(b)]. Such a path has no counterpart in the single chain system and it cannot be continuously deformed into a loop that surrounds an isolated critical point. Nevertheless, we argue that the Chern number associated with this path is determined by the topological character of the HI-MI critical point. A simple way to approach this problem is to note that two loops, each encircling one of the two branches, can be deformed into a single loop which encircles both branches. Such a loop corresponds to pumping of two bosons as discussed above. Therefore, by symmetry of the $\pm\lambda$ branches, each of the isolated loops entails pumping of one boson along the ladder, or half a boson per chain.

The distribution of quantized charge among different loops in parameter space can be succinctly represented in terms of a fictitious quantized magnetic flux running through gapless regions in the three-dimensional parameter space. Two flux quanta, one for each chain, are inserted through the isolated HI-MI critical point in the $t_\perp$ direction, and must split evenly between the two branches at $\pm\lambda$. The quantized pumping therefore defines a topological index, the fictitious quantized flux, that is associated with the gapless phases.

More than two chains.—Without interchain coupling, N parallel chains are just N copies of the single chain problem, and so an adiabatic cycle around the critical point implies pumping of N bosons along the decoupled chains. As before, this charge cannot change suddenly with the introduction of interchain coupling $t_\perp$. Hence, in the extended parameter space the critical point at the origin $t_\perp = 0$ is a source of N quanta of the fictitious flux. The gapless phase at finite $t_\perp$ may branch out, as in the case of two chains, while the fictitious flux running through all the branches must add up to exactly N.

There is another topological constraint on the branching of the gapless phase with increasing $t_\perp$. From the construction of Refs. [18,19], follows a sharp distinction between the Haldane insulator and the Mott insulator phase on any ladder with an odd number of chains. That is, without breaking inversion symmetry, $2N + 1$ decoupled chains in the HI phase cannot be connected adiabatically to decoupled chains in the MI phase by an adiabatic path going through finite $t_\perp$, in contrast to the two leg case considered above. Therefore in a ladder with odd number of legs the gapless phase must persist indefinitely on the plane with inversion symmetry, i.e, $\lambda = 0$.

Conclusions.—Topological properties of matter are usually associated with gapped regions of the phase diagram. Here, we have shown that in a model of interacting bosons, it is natural to associate a topological "flux" with the gapless (superfluid) regions, which is defined by the pumped charge upon encircling these regions adiabatically. This property can be argued to be more profound, in the sense that the gapped phases discussed here are only distinct from each other as long as certain symmetries (e.g., inversion symmetry) are preserved, while the topological flux associated with the gapless region is robust against arbitrary particle number conserving perturbations. This principle can be used to impose constraints on the topology of the phase diagram; for example, it implies that a gapless region which carries a nonzero topological flux cannot terminate.

Similarly, topological insulators in two and three dimensions are only well-defined as long as time-reversal symmetry is preserved. However, the gapless region separating the topologically trivial and nontrivial phase may carry a topological flux, which remains well defined even when time-reversal symmetry is broken. That can hopefully shed new light on the nature of topological insulators [23].

This work was supported in part by NSF under Grants No. DMR-0705472 and No. DMR-0757145 (E. B.), the US Israel BSF (EA and EB), ISF (E. A.), and the Harvard Society of Fellows (M. L.). The calculations were run on the Odyssey cluster supported by the FAS Sciences Division Research Computing Group. E. B. and E. A. thank the Aspen Center for Physics, where part of this work was done.

[1] D. J. Thouless, Phys. Rev. B **27**, 6083 (1983).
[2] Q. Niu and D. J. Thouless, J. Phys. A **17**, 2453 (1984).
[3] J. E. Avron and R. Seiler, Phys. Rev. Lett. **54**, 259 (1985).
[4] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).
[5] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science **314**, 1757 (2006).
[6] J. E. Moore and L. Balents, Phys. Rev. B **75**, 121306 (2007).
[7] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).
[8] M. Konig, S. Wiedmann, C. Brune, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science **318**, 766 (2007).
[9] D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan, Nature (London) **452**, 970 (2008).
[10] L. Fu and C. L. Kane, Phys. Rev. B **74**, 195312 (2006).
[11] R. Shindou, J. Phys. Soc. Jpn. **74**, 1214 (2005).
[12] G. Pupillo, A. Micheli, M. Boninsegni, I. Lesanovsky, and P. Zoller, Phys. Rev. Lett. **104**, 223002 (2010).
[13] J. Sebby-Strabley, M. Anderlini, P. S. Jessen, and J. V. Porto, Phys. Rev. A **73**, 033605 (2006).
[14] S. Fölling, S. Trotzky, P. Cheinet, M. Feld, R. Saers, A. Widera, T. Müller, and I. Bloch, Nature (London) **448**, 1029 (2007).
[15] E. G. Dalla Torre, E. Berg, and E. Altman, Phys. Rev. Lett. **97**, 260401 (2006).
[16] E. Berg, E. G. Dalla Torre, T. Giamarchi, and E. Altman, Phys. Rev. B **77**, 245119 (2008).
[17] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009).
[18] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, arXiv:0909.4059.
[19] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010).
[20] See supplemental material at http://link.aps.org/supplemental/10.1103/PhysRevLett.106.110405 for a proof of the quantized pumping using fermionization and additional numerical data.
[21] E. Altman and A. Auerbach, Phys. Rev. Lett. **89**, 250404 (2002).
[22] H. J. Schulz, Phys. Rev. B **34**, 6372 (1986).
[23] Z. Ringel and E. Altman (unpublished).