Non-Abelian Operations on Majorana Fermions via Single-Charge Control

Karsten Flensberg

Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA
and Niels Bohr Institute, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark

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We demonstrate that non-Abelian rotations within the degenerate ground-state manifold of a set of Majorana fermions can be realized by the addition or removal of single electrons, and propose an implementation using Coulomb blockaded quantum dots. The exchange of electrons generates rotations similar to braiding, though not in real space. Unlike braiding operations, rotations by a continuum of angles are possible, while still being partially robust against perturbations. The quantum dots can also be used for readout of the state of the Majorana system via a charge measurement.

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Elementary excitations of topological materials can have unusual properties, such as statistics different from that of fermions or bosons. The most interesting possibility is non-Abelian statistics which is believed to be realized, for example, in the $5/2$ fractional quantum Hall system and in topological superconductors, where the low-energy quasiparticles are Majorana fermions [1]. Following Kitaev [2], recent theoretical achievements have shown that topological superconductors should be realizable in semiconductor or metallic systems with the right combination of spin-orbit coupling, induced superconductivity and applied magnetic field [3–12], which has brought new optimism into the search. Several methods to observe the non-Abelian nature of particle exchange have been suggested. For fractional quantum systems, interferometry of paths [13,14] and Coulomb blockade peak spacings [15] sensitive to the number of enclosed quasiparticles have been analyzed, and the former recently investigated experimentally [16]. The nonlocal nature combined with Coulomb interactions has also been exploited theoretically [17]. For semiconductor wire systems, exchange of particles in one-dimensional network structures was recently proposed as a direct way to observe non-Abelian features of Majorana bound states [18].

Moreover, non-Abelian quasiparticles have been suggested as a basis for topological quantum computing [19], computational steps being done by physically exchanging (braiding) positions of the quasiparticles, thus performing a unitary rotation in the ground-state manifold. In principle, the exchange operation only depends on the topology of the exchange and it is therefore argued to be robust against local perturbations [2,13]. However, braiding of Majorana fermions must be supplemented with nonprotected gates to give a universal set of operations. Several methods have been suggested, including merging the quasiparticles for a certain time [20] or combining topological and conventional qubits [21–23].

In this Letter, a different method for performing operations on a set of Majorana bound states is analyzed. The individual operations are adiabatic tunnel processes of a single electron from a Coulomb blockaded quantum dot coupled to one or two Majorana modes. The processes result in rotations of the ground-state manifold, reminiscent of actual physical exchange of the particles. Like real-space braidings, the tunnel-braid operations are robust against dephasing. However, they are sensitive to electrical noise on the tunnel amplitudes, and in this respect not topologically protected. The electrical control, on the other hand, allows rotations with tunable angle. Moreover, to achieve the maximal protection of the tunnel braids, a certain phase difference (controllable by flux) between the two Majorana bound states is needed.

The system we have in mind is a connected network of one-dimensional wires, made of out of semiconducting nanowires, or gated two-dimensional electron gas, with spin-orbit coupling, induced superconductivity and applied magnetic field, and in addition a number of quantum dots, see the sketch in Fig. 2(a). For parameters where the superconductor is in the topological phase, Majorana bound states (MBSs) exist at the ends of the wires [6–9]. They are zero energy solutions to the Boguliubov–de Gennes equations and have the general form

$$\gamma_i = \int d\mathbf{r}(f_i\Psi_\uparrow + f_i^*\Psi_\uparrow^\dagger + g_i\Psi_\downarrow + g_i^*\Psi_\downarrow^\dagger). \qquad (1)$$

The MBS operators obey $\gamma_i = \gamma_i^\dagger$ and $\gamma_i^2 = 1$. Here f_i and g_i are functions of the spatial coordinate $\mathbf{r}$ and $\Psi_\sigma = \Psi_\sigma(\mathbf{r})$ is the electron field operator. The Majorana bound states are tunnel coupled to quantum dots in the Coulomb blockade regime, i.e., the charge state on a dot is restricted to be either N or $N+1$ electrons, with N some arbitrary number. The level spacing of a dot is supposed to be much larger than the temperature, so that only one quantum state is involved. Furthermore, because a large magnetic field is applied to induce the topological superconducting phase, it is assumed that spin degeneracy of the dots is broken, so that we only have to consider a single spin direction, say, spin-up. The dot potential energies can be controlled by electrostatic gates.

Projection onto Majorana bound states $\{\gamma_i\}$ coupled to an electron level (described by a spin-up electron annihilation operator c) leads to an effective low-energy tunnel Hamiltonian

$$H_T = \sum_i (v_i c - v_i^* c^\dagger)\gamma_i, \quad (2)$$

where v_i is the Hamiltonian overlap between the electron wave function, ϕ, and the spin-up component of the MBS $v_i = \langle f_i|H|\phi\rangle$. Later it is assumed that v_i is controllable.

First, we consider the situation with a single MBS γ_1 coupled to a single dot and study the transition of the ground state as the occupancy of the dot is changed adiabatically. Two Majorana fermions are needed to define a usual (Dirac) fermion. Using Majorana states γ_1 and γ_2 as the basis, a fermion M12 with annihilation operator $d = (\gamma_1 + i\gamma_2)/2$ is defined. The two eigenstates of $d^\dagger d$ are denoted $|0\rangle_{\mathrm{M12}}$ and $|1\rangle_{\mathrm{M12}}$. The Hamiltonian for a dot level c connected to a Majorana mode $\gamma_1 = d + d^\dagger$ is then

$$H_1 = \varepsilon c^\dagger c + (v_1^* c^\dagger - v_1 c)(d + d^\dagger), \quad (3)$$

where ε is the dot level energy, measured relative to the chemical potential of the superconductor. The Hilbert space contains 4 states, and the Hamiltonian is block diagonal by conservation of the total parity. For both even and odd parity the Hamiltonian matrix is

$$H_{1,\text{even or odd}} = \begin{pmatrix} 0 & v_1 \\ v_1^* & \varepsilon \end{pmatrix}, \quad (4)$$

with the basis for even and odd total parity cases being $\{|0\rangle_{\mathrm{D}}|0\rangle_{\mathrm{M12}}, |1\rangle_{\mathrm{D}}|1\rangle_{\mathrm{M12}}\}$ and $\{|0\rangle_{\mathrm{D}}|1\rangle_{\mathrm{M12}}, |1\rangle_{\mathrm{D}}|0\rangle_{\mathrm{M12}}\}$, where 0 (1) represent an empty (full) fermion state, and $|\cdots\rangle_{\mathrm{D}}$ denotes the dot state.

Now an operation P_1 that adiabatically changes ε/v_1 from $-\infty$ to $+\infty$ is defined, see Fig. 1 [24]. If the original state of the Majorana system is $|i\rangle_{\mathrm{M}} = \alpha|0\rangle_{\mathrm{M12}} + \beta|1\rangle_{\mathrm{M12}}$ and the dot state is $|1\rangle_{\mathrm{D}}$, the state of the dot plus Majorana system is at any gate potential ε given by the superposition (up to an overall dynamical phase factor)

$$|\psi\rangle = a(\varepsilon)|1\rangle_{\mathrm{D}}(\alpha|0\rangle_{\mathrm{M12}} + \beta|1\rangle_{\mathrm{M12}}) + b(\varepsilon)|0\rangle_{\mathrm{D}}(\alpha|1\rangle_{\mathrm{M12}} + \beta|0\rangle_{\mathrm{M12}}), \quad (5)$$

where $v_1 b(\epsilon) = Ea(\varepsilon)$ and $E = \varepsilon/2 - \sqrt{(\varepsilon/2)^2 + v_1^2}$. The dot and the Majorana system is thus entangled during the operation, but not at the end of the operation where $\varepsilon/v_1 \to \infty$ and $b \to 1$. Therefore, the operation P_1 generates an inversion of the occupation of the M12 fermion, which in basis independent notation [25] is expressed as

$$P_1 : |i\rangle_{\mathrm{M}} \mapsto \gamma_1|i\rangle_{\mathrm{M}}. \quad (6)$$

If a number of Majorana bound states are connected to quantum dots, it is thus possible to manipulate the Majorana system by repeated applications of P operations,

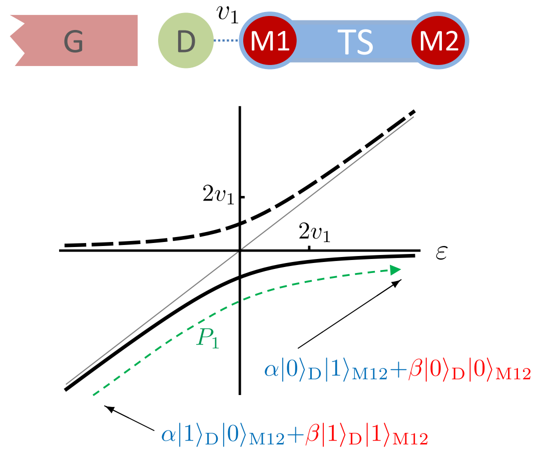

FIG. 1 (color online). A single Majorana state (M1) coupled to a quantum dot (D) by a tunnel coupling v_1. The Majorana state M1 is combined with another Majorana bound state, M2, to form a fermion, M12. Starting in the ground state, the operation P_1 takes the energy ε [using the gate (G)] from negative to positive values causing an adiabatic transition from a full dot to an empty dot. The electron is added to the superconductor, thus inverting the parity of the Majorana system. Because the total even (red [dark gray]) and odd (blue [light gray]) parity states are degenerate the inversion is independent of the details of the transition.

yielding a new state $\gamma_1, \ldots, \gamma_m|i\rangle_{\mathrm{M}}$. However, because $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$ this only provides a finite number of operations. Interestingly, two consecutive operations P_1P_2 correspond to a real-space process where one Majorana bound state is rotated around the other [26].

A natural question is how sensitive the operations P_i are to decoherence of the dot charge. To investigate this, we add environmental degrees of freedom that couple to the dot charge. At beginning of the P_1 process the initial state is $|\Psi_i\rangle = |1\rangle_{\mathrm{D}}|i\rangle_{\mathrm{M}}|n\rangle_{\mathrm{env}}$, where $|n\rangle_{\mathrm{env}}$ is the initial environment state. When expanding in powers of the tunneling Hamiltonian, the time-evolution operation can be collected in powers of γ_1 as $U(T,0) = U_0 + \gamma_1 U_1$. If the transition is done slowly enough to guarantee that the occupancy of the dot changes by one (can be checked by measuring the charge on the dot) only the part $\gamma_1 U_1$ survives, and Eq. (6) holds, regardless of the dot's coupling to the environment.

As mentioned, the single MBS processes in Eq. (6) only allow a limited set of operations. A richer set is possible if the dots are connected to *two* Majorana modes, as in Fig. 2(a). Using the same basis as above, the even and odd sector Hamiltonians for MBS γ_1 and γ_2 coupled to a dot (D1 in Fig. 2) are in this case

$$H_{12,\text{even or odd}} = \begin{pmatrix} 0 & v_{\text{even or odd}} \\ v^*_{\text{even or odd}} & \varepsilon \end{pmatrix}, \quad (7)$$

where $v_{\text{even (odd)}} = v_1 - (+)iv_2$. In general, the matrix elements v_{even} and v_{odd} are different and the degeneracy between the even and the odd cases is lifted. The ground-state energies are [see Fig. 2(b)]

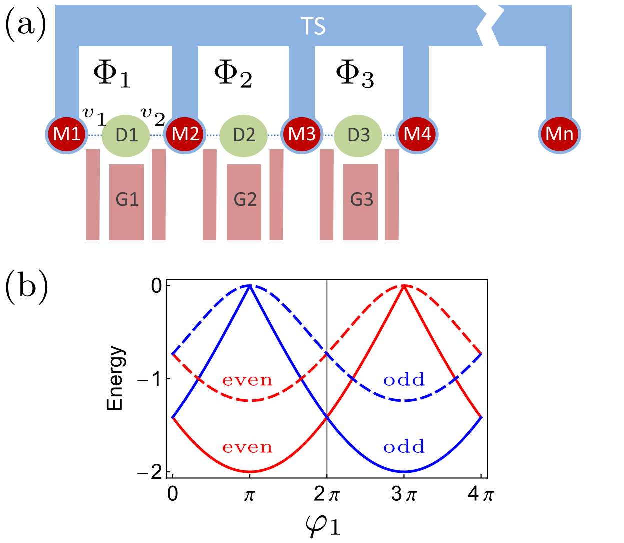

FIG. 2 (color online). (a) A one-dimensional array of Majorana states (M1, ..., Mn) coupled to quantum dots (D1, D2,...) in the Coulomb blockade regime. Each dot is tunnel coupled to two Majorana states with tunnel barriers [controlled by the gates adjacent to the plunger gates (G1, G2,...)]. Changing the occupancy of the dots by one electron creates the unitary rotations P_{ij}. (b) The ground-state energy of one dot coupled to two Majorana bound states, with $|v_1| = |v_2|$ for even (red) and odd (blue) total parity of a dot and its two connecting MBSs. Even and odd cases are degenerated for $\varphi_1 = 2n\pi$, which makes the P_{ij} operations partially protected. The full and dashed lines are for $\varepsilon/|v_1| = 0$ and 2, respectively.

$$E_{\text{even or odd}} = \varepsilon/2 - \sqrt{(\varepsilon/2)^2 + v^2 \mp 2|v_1 v_2| \sin(\varphi_1/2)}, \tag{8}$$

where $v^2 = |v_1|^2 + |v_2|^2$ and $\varphi_1 = 2\text{Arg}(v_1/v_2)$. The phase difference is controlled by the flux Φ_1 and up to constant phase shift, we can write $\varphi_1 = \Phi_1/\Phi_0$, with Φ_1 being the flux in loop 1 and $\Phi_0 = h/2e$. Note that Eq. (8) is 4π periodic in φ_1 [27].

Again, consider an adiabatic process that transfers an electron from a dot but this time to the two Majorana states γ_1 and γ_2. Unlike the case with a single MBS, the resulting rotation of the Majorana system is in general not independent of the time spend in the adiabatic process, because of the energy difference in Eq. (8), see Fig. 2(b). The degeneracy is restored only when the phase difference is $\varphi_1 = 2n\pi$ (n integer), which therefore requires tuning the magnetic flux Φ_1 [Fig. 2(a)]. At this degeneracy point, v_1/v_2 is real which allows the Hamiltonian to be written as

$$H_{12} = \varepsilon\tilde{c}^\dagger\tilde{c} + v(\tilde{c}^\dagger - \tilde{c})\gamma_{12}, \tag{9}$$

where a new Majorana operator is defined

$$\gamma_{12} = \frac{1}{v}(|v_1|\gamma_1 + |v_2|\gamma_2), \tag{10}$$

and where a common phase is absorbed into the dot-electron operator $\tilde{c} = c\exp[i\text{Arg}(v_1)]$. Thus, since the Hamiltonian (9) has the same form as (3), a dot coupled to two MBSs reduces (at the degeneracy point) to a dot coupled to a single Majorana state γ_{12}. The conclusion from above therefore also carries over: by adiabatically changing the electron number of the dot, the following rotation is performed

$$P_{12}: |i\rangle \mapsto \gamma_{12}|i\rangle. \tag{11}$$

To understand the rotations that can be generated by repeated applications of P_{12} (with different ratios $|v_1/v_2|$), we use the following Pauli matrixes acting on the two level system spanned by γ_1 and γ_2: $\sigma_x = \gamma_1$, $\sigma_y = \gamma_2$, and $\sigma_z = -i\gamma_1\gamma_2$. In this language, the operation P_{12} makes a π rotation around an axis in the x-y plane, but other rotation angles around lines in the x-y plane cannot be done. In contrast, when applying a pair $P_{12}P'_{12} = (u\gamma_1 + v\gamma_2)(u'\gamma_1 + v'\gamma_2) = (uu' + vv') + i(uv' - vu')\sigma_z$ a rotation around the z axis with tunable angle is performed. A braid operation also rotates around the z axis, but by an angle restricted to $\pi/2$. Instead, using four MBSs and the even-parity subspace to define a qubit [20], a universal set of single qubit rotations is in fact generated by pairs of P operators. Again, $P_{12}P'_{12}$ is a rotation around the z axis [in the basis $\{(00), (11)\}$ defined below], whereas $P_{23}P'_{23}$ now gives a rotation around the x axis with a controllable angle [28].

A special and illuminating case is when the dots couple with equal strength to two MBSs [$|v_1| = |v_2|$ in Eq. (10)], which results in operators $F_i = \frac{1}{\sqrt{2}}(\gamma_i + \gamma_{i+1})$ acting on nearest neighbors. They are related to braid operators $B_i = \frac{1}{\sqrt{2}}(1 + \gamma_{i+1}\gamma_i)$ [26] by $B_i = F_i\gamma_i = \gamma_{i+1}F_i$. The F_i operators fulfill $F_i^2 = 1$ and

$$F_iF_j = -F_jF_i, \qquad |i - j| > 1 \tag{12a}$$

$$F_iF_{i+1}F_i = -F_{i+1}F_iF_{i+1}, \tag{12b}$$

which differs by a minus sign from the relations defining the braid group [1]. As a side remark, F_i form a projective representation of the permutation group [29].

To demonstrate the non-Abelian nature of the tunnel-braid operations, consider now an explicit example with four Majorana states and three dots. The state of the superconductor is initialized by tuning the dots and the magnetic field to fuse Majorana pairs (1, 2) and (3, 4) and letting them relax. The initial state is $|00\rangle = |0\rangle_{M12}|0\rangle_{M34}$, referring to the occupation of the fermions, $d_1 = (\gamma_1 + i\gamma_2)/2$ and $d_2 = (\gamma_3 + i\gamma_4)/2$. We will consider applications of pairs of F_i and hence restrict to the subspace of even parity, spanned by $|00\rangle$ and $|11\rangle = d_2^\dagger d_1^\dagger|00\rangle$. The possible unitary transformations are given by

$$(F_1F_2)_{\text{even}} = [(F_2F_3)_{\text{even}}]^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \tag{13}$$

and $(F_1F_3)_{\rm even} = \sigma_x$ (up to phase factors). Other permutations can be deduced from $F_iF_j = [F_jF_i]^{-1}$. Application of F_1F_2 or F_2F_1 gives

$$F_1F_2|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{14a}$$

$$F_2F_1|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle). \tag{14b}$$

For one sequence the resulting state is an eigenstate of σ_x and for the other it is an eigenstate of σ_y. However, the expectation values of neither of these operators are easily measured using quantum dots, capable of measuring charge only. Initial application of F_2F_3 and F_3F_2 in Eqs. (14) rotates to states that can be distinguished by an occupation measurement and this yields

$$F_1F_2F_2F_3|00\rangle = |11\rangle, \tag{15a}$$

$$F_2F_1F_3F_2|00\rangle = |00\rangle. \tag{15b}$$

The last sequence can be implemented in the following way: the potentials on the dots D1, D2, and D3 in Fig. 2(a) are successively increased from positive to negative voltages, thus emptying the dots, and then D2 is filled again by tuning the D2 potential back to positive voltages. The sequence (15a) is done in the same way, except in a different order.

Finally, we discuss how the state of the Majorana system can be read out. Several methods have been proposed to measure the state of coupled Majorana modes, including observing changes in the current-phase relation of a Josephson junction with two Majorana states in the loop [3,30], or by interferometry [1,14]. In the present setup it is natural to use the quantum dots, which is indeed possible. By choosing the phase difference between two MBSs, so that the fused even or odd states are split in energy, the occupancy can be read from the adiabatic curves similar to the one in Fig. 1, because the off-diagonal elements in Eq. (7), unlike in Eq. (4), differ for the even and odd cases. Maximal visibility is achieved for $\varphi_1 = \pi$, where the two off-diagonal matrix elements are $|v_1| + (-)|v_2|$ for even (odd), respectively.

In conclusion, a scheme for manipulating the state of a set Majorana fermions has been proposed. It allows for demonstration of the non-Abelian nature of the quasiparticles, albeit not by real-space exchanges. Instead the exchanges are performed via removal or addition of real electrons, made possible by Coulomb blockade.

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