Generalized Basset-Boussinesq-Oseen Equation for Unsteady Forces on a Sphere in a Compressible Flow

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Viscous compressible flow around a sphere is considered in the limit of zero Reynolds and Mach numbers. An exact expression for the force on the sphere undergoing arbitrary motion with compressibility effects is presented. Quasisteady, inviscid-unsteady, and viscous-unsteady force components are identified. Numerical results are in excellent agreement with the theory. The present formulation offers an explicit expression for the unsteady force in the time domain and can be considered as a generalization of the Basset-Boussinesq-Oseen equation to compressible flow.

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Introduction.—The unsteady force on a particle in accelerated motion was first analyzed by Stokes [1]. Later Basset [2], Boussinesq [3], and Oseen [4] independently examined the time-dependent force on a sphere in a quiescent viscous incompressible fluid. The resulting equation of motion the so-called BBO equation, can be written as

$$m_p \frac{d\mathbf{v}}{dt} = -6\pi a \mu \mathbf{v} - \frac{m_f}{2}\frac{d\mathbf{v}}{dt} - 6a^2\rho\sqrt{\pi\nu} \times \int_{-\infty}^{t} \frac{1}{\sqrt{t-\xi}} \frac{d\mathbf{v}}{dt}\Big|_{t=\xi} d\xi, \quad (1)$$

where m_p, $\mathbf{v}(t)$, and a are the particle mass, velocity, and radius. ρ, m_f, μ, and ν are the density, displaced-mass, dynamic viscosity, and kinematic viscosity of the fluid. The three terms on the right-hand side are the quasisteady (Stokes) drag, inviscid-unsteady (added-mass), and viscous-unsteady (Basset history) forces, respectively. The BBO equation has been extended to nonuniform creeping flows by Maxey and Riley [5] and Gatignol [6].

Our primary goal is to extend the BBO equation to compressible flows. The first work relevant to our goal appears to be that of Zwanzig and Bixon [7] (also see Metiu *et al.* [8]). Temkin and Leung [9] and Guz [10] have presented solutions that are essentially identical.

The purpose of this work is to present an explicit expression for the time-dependent force on a spherical particle undergoing arbitrary unsteady motion on the acoustic time scale such that compressibility effects are important. Attention is restricted to the zero Reynolds- and Mach-number limits. We use previously derived solutions of the linearized compressible Navier-Stokes equations in the Fourier or Laplace domains to determine the force on a particle in response to a delta-function acceleration in the time domain. This force response is then used to construct an expression for the time-dependent force on a particle undergoing arbitrary motion. The resulting expression can be interpreted as the generalization of the BBO equation to compressible flows. We show that compressibility causes the inviscid-unsteady force to assume an integral representation first derived by Longhorn [11]. We obtain the effect of compressibility on the viscous-unsteady force. The results are compared with numerical simulations.

Problem formulation.—We consider the unsteady motion of a particle in a quiescent compressible Newtonian fluid. We consider the limit of Re → 0 and M → 0 such that the perturbation field generated by the particle motion is governed by the linearized compressible Navier-Stokes equations. Here, M and Re are suitably defined Mach and Reynolds numbers. The continuity and momentum equations reduce to the form given by Zwanzig and Bixon [7],

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' = 0, \quad (2)$$

$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} + \nabla p' - \mu \nabla^2 \mathbf{u}' - \left(\mu_b + \frac{1}{3}\mu\right)\nabla\nabla \cdot \mathbf{u}' = 0. \quad (3)$$

In Eqs. (2) and (3), properties associated with the quiescent fluid are denoted by the subscript 0, perturbation quantities are denoted by the superscript $'$, $\mathbf{u}$ is the velocity, p is the pressure, and μ_b is the bulk viscosity. Speed of sound

$$c_0 = \sqrt{(\partial p/\partial \rho)_s} = \sqrt{p'/\rho'} \quad (4)$$

is used as a closure relation. These linearized equations have been solved analytically by Zwanzig and Bixon [7], who obtained an explicit expression for the force on the particle in the frequency domain. Given a general particle motion with velocity $v(t)$, the solution of Eqs. (2)–(4) in Laplace space can be written as

$$\mathcal{F}(s) = -m_f s G(r_1, r_2) \mathcal{L}(v) \quad (5)$$

where $\mathcal{F}(s) = \mathcal{L}(F(t))$ and $\mathcal{L}(v)$ are the Laplace transforms of the time-dependent force $F(t)$ and rectilinear particle velocity $v(t)$, respectively, and $m_f = 4\pi\rho_0 a^3/3$. The transfer function $G(r_1, r_2)$ is given by

$$G(r_1, r_2) = \frac{(9 + 9r_1 + 2r_1^2)(1 + r_2) + (1 + r_1)r_2^2}{r_1^2(1 + r_2) + (2 + 2r_1 + r_1^2)r_2^2}, \quad (6)$$

$$r_1(s) = \frac{as/c_0}{\sqrt{1 + (\mu_b/\mu + 4/3)\nu s/c_0^2}}, \quad r_2(s) = a\sqrt{\frac{s}{\nu}}. \quad (7)$$

Solution for impulsive motion.—Since the problem is linear, the force on a particle undergoing arbitrary rectilinear motion $v(t)$ can be expressed as

$$F(t) = \int_{-\infty}^{t} F_\delta(t - \xi) \frac{dv}{dt}\Big|_{t=\xi} d\xi, \quad (8)$$

where $F_\delta(t)$ is the force response to a delta-function acceleration (i.e., corresponding to a unit step change in particle velocity). Using Eq. (5), $F_\delta(t)$ can be expressed as

$$\mathcal{F}_\delta(s) = -m_f G(r_1, r_2). \quad (9)$$

An explicit Laplace inverse transform of Eq. (9) and, therefore, a closed-form expression for $F_\delta(t)$ is not readily available. Before constructing the time-domain solution, we first analyze the limiting case of incompressible flow by letting $c_0 \to \infty$ to obtain in the time domain

$$F_{\delta,\mathrm{inc}}(t) = -6\pi a\mu H(t) - \frac{m_f}{2}\delta(t) - 6a^2\rho_0\sqrt{\frac{\pi\nu}{t}}H(t), \quad (10)$$

where $H(t)$ is the Heaviside step function.

Compressibility effect on inviscid-unsteady force.—We isolate the three terms (quasisteady, inviscid-unsteady, and viscous-unsteady forces) on the right-hand side of Eq. (1) and investigate the effect of compressibility. First, we consider the compressibility effect on the inviscid-unsteady force. The inviscid limit is obtained by substituting $\nu = 0$ in Eq. (9) to get in the time domain

$$F_{\delta,\mathrm{iu}}(\tau) = -m_f \frac{c_0}{a} e^{-\tau} \cos\tau H(\tau), \quad (11)$$

where $\tau = c_0 t/a$. The effect of compressibility on the inviscid-unsteady force can be established by comparing Eq. (11) with the second term on the right-hand side of Eq. (10). The finite speed of sound destroys the instantaneous relationship between acceleration and force by regularizing the singular delta-function kernel to a smooth oscillatory exponential decay. From a physical perspective, this can be explained by the compression and rarefaction waves that emanate from the accelerated particle which propagate outward at finite speed. However, due to the exponential-decay term in Eq. (11), the compressibility effect is significant only for $\tau \lesssim 10$.

The above inviscid-unsteady force was first obtained by Longhorn [11] and is valid only in the zero Mach-number limit. The right-hand side of Eq. (11) can be considered to be the response kernel for a delta-function acceleration for $M \to 0$. Note that $\int_0^\infty e^{-\tau} \cos\tau d\tau = 1/2$, and thus over times much longer than the acoustic time scale, the net impulse on the particle reduces to the correct limit of the incompressible added-mass force. The corresponding kernels for finite Mach numbers can be obtained through numerical simulations, see Parmar *et al.* [12].

Asymptotic behaviors of compressible viscous-unsteady force.—We now examine the effect of compressibility on the viscous-unsteady force. To study the force at arbitrary times, we resort to numerical inversion of Eq. (9). With V denoting the scale of the velocity variation, we define the Reynolds and Mach numbers as $\mathrm{Re} = \rho_0 V a/\mu$ and $\mathrm{M} = V/c_0$. We can write

$$F_\delta(\tau)/(m_f c_0/a) = -\mathcal{L}^{-1}(G(R_1, R_2)), \quad (12)$$

where $\mathcal{L}^{-1}$ denotes the Laplace inverse with respect to the nondimensional time $\tau = c_0 t/a$ and R_1 and R_2 are functions of non-dimensional Laplace variables $S = as/c_0$ and modified Knudsen number $\mathrm{Kn}' = \mathrm{M}/\mathrm{Re}$. For the continuum assumption we are interested in $\mathrm{Kn}' \lesssim O(10^{-2})$.

Four different asymptotic regimes can be identified: (i) Regime I: Very short time, defined as $\tau \ll \mathrm{Kn}' \ll 1$, (ii) Regime II: Intermediate short time, defined as $\mathrm{Kn}' \ll \tau \ll 1$, (iii) Regime III: Intermediate long time, defined as $1 \ll \tau \ll 1/(\mathrm{ReM})$, (iv) Regime IV: Very long time, defined as $1 \ll 1/(\mathrm{ReM}) \ll \tau$.

The very short time behavior of $F_\delta(\tau)$ can be obtained by considering $1 \ll \mathrm{Kn}'|S| \ll |S|$. The corresponding time-domain force response in Regime I is

$$F_\delta(\tau) \sim -\left(\frac{4}{9} + \frac{2}{9}\sqrt{\frac{\mu_b}{\mu} + \frac{4}{3}}\right) 6a^2\rho_0 c_0 \sqrt{\frac{\pi \mathrm{Kn}'}{\tau}} H(\tau). \quad (13)$$

Comparing with the third term on the right-hand side of Eq. (10), which can be written as $-6a^2\rho_0 c_0\sqrt{\pi\mathrm{Kn}'/\tau}$, it can been seen that compressibility modifies the viscous-unsteady force at very short times by a factor that depends on μ_b/μ. For $\mu_b = 0$, compressibility reduces the unsteady force by $4(1 + 1/\sqrt{3})/9 \approx 0.70$.

The intermediate short time behavior is obtained by considering $\mathrm{Kn}'|S| \ll 1 \ll |S|$. Then $G(R_1, R_2)$ can be simplified and in Regime II we obtain

$$F_\delta(\tau) \sim -m_f \frac{c_0}{a} e^{-\tau} \cos\tau - \frac{8}{3} a^2\rho_0 c_0 \sqrt{\frac{\pi \mathrm{Kn}'}{\tau}} H(\tau). \quad (14)$$

The first term is same as $F_{\delta,\mathrm{iu}}(\tau)$ given by Eq. (11). Comparing the second term with the third term on the right-hand side of Eq. (10), it can been seen that the viscous-unsteady force at intermediate short times is reduced by a factor of $4/9 \approx 0.44$ because of compressibility. Note that this reduction is independent of μ_b/μ. As will be seen below in Fig. 1, Regime II can be observed only if $\mathrm{Kn}' \ll 1$. With increasing Kn', the duration of the Regime II reduces and vanishes entirely for $\mathrm{Kn}' \approx 10^{-2}$.

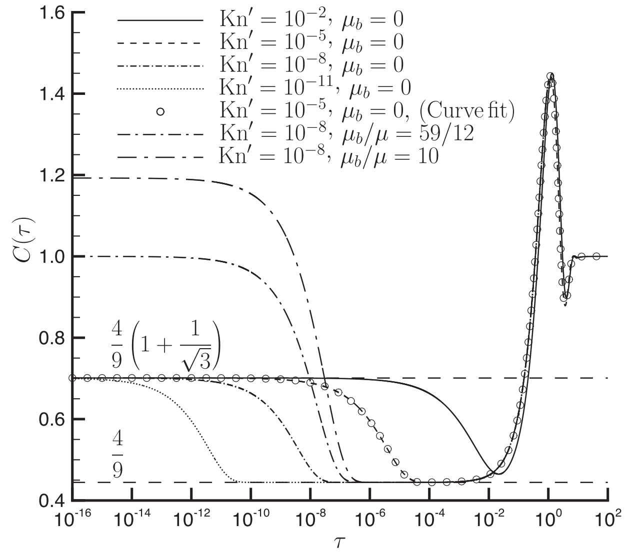

FIG. 1. The behavior of $C(\tau)$ that accounts for the compressibility effect on the viscous-unsteady force.

The intermediate long-time behavior can be obtained by considering $|S| \to 0$ and carrying out the Laplace inverse to obtain in Regime III

$$F_\delta(\tau) \sim -6\pi a \mu H(\tau) - 6a^2\rho_0 c_0 \sqrt{\frac{\pi \mathrm{Kn}'}{\tau}}. \tag{15}$$

Comparing with Eq. (10), both the quasisteady and the viscous-unsteady forces are recovered and found to be unaffected by compressibility effects. Strictly speaking, the above linear solution is valid for $\tau \gg 1$ and the additional limit of $\tau \ll 1/(\mathrm{ReM})$ arises only from the neglect of the nonlinear terms. In deriving the linearized equations, the assumption that the inertial terms are negligible compared to the viscous terms implies that the length scale $L \ll \nu/V$. If we take the length scale to grow by diffusion as $\sqrt{\nu t}$, the linearization can be justified only for $t \ll \nu/V^2$. Expressed in terms of the acoustic time scale, this restriction becomes $\tau \ll 1/(\mathrm{ReM})$. Note that the above argument applies in an incompressible flow also, and the nonlinear effects can be shown to become important for $\tau_c \gg 1/\mathrm{Re}$, where τ_c is time nondimensionalized by the convective time scale a/V. This is consistent with past results for incompressible flow that the Basset history kernel is valid only for $\tau_c \ll 1/\mathrm{Re}$ even at low Reynolds numbers (see Mei and Adrian [13]). Thus, the very long time force behavior in Regime IV will depend on both Re and M in a complex manner.

Numerical evaluation of viscous-unsteady force.—We isolate the viscous-unsteady force from the overall force expression given in Eq. (12) by subtracting the quasisteady contribution and the inviscid-unsteady force given in Eq. (11).

$$\frac{F_{\delta,\mathrm{vu}}(\tau)}{m_f c_0/a} = -\left(\mathcal{L}^{-1}(G) - \frac{9}{2}\,\mathrm{Kn}' - e^{-\tau}\cos\tau\right). \tag{16}$$

We recast this viscous-unsteady response to delta-function acceleration in the following form

$$F_{\delta,\mathrm{vu}}(\tau)/(m_f c_0/a) = -\frac{9}{2}\sqrt{\mathrm{Kn}'/(\pi\tau)}\,C(\tau), \tag{17}$$

where $C(\tau)$ is a compressible correction function, defined as the ratio of $F_{\delta,\mathrm{vu}}(\tau)$ relative to the incompressible form of the viscous-unsteady force.

Figure 1 shows $C(\tau)$ plotted against τ. In Regime I ($\tau \ll \mathrm{Kn}' \ll 1$) the correction function approaches 0.70 at very short times. Also, we observe that $C(\tau) \to 1$ as $\tau \to \infty$ as expected. At intermediate short times (which exist only for $\mathrm{Kn}' \ll 10^{-2}$) the correction function takes a constant value of 0.44. At about $\tau = O(10^{-2}\mathrm{Kn}')$, $C(\tau)$ starts to deviate from its limiting value of 0.70 and monotonically decreases to 0.44 at about $\tau = O(\mathrm{Kn}')$. The transition from Regime II to Regime III that occurs at $\tau \approx O(1)$ is more complex. At $\tau = O(10^{-2})$, $C(\tau)$ increases rapidly irrespective of Kn' toward a peak value of about 1.45 before decreasing in a strongly damped oscillatory manner. Thus, the compressibility correction to the viscous-unsteady force is bounded between 0.44 and 1.45 for $\mu_b = 0$. The sensitivity of $C(\tau)$ to the bulk viscosity is also shown in Fig. 1.

Unsteady force kernels.—Based on results presented in the previous sections, we write

$$F_\delta = F_{\delta,\mathrm{qs}} + F_{\delta,\mathrm{iu}} + F_{\delta,\mathrm{vu}}, \tag{18}$$

where $F_{\delta,\mathrm{qs}} = -6\pi\mu a H(\tau)$ is the quasisteady force in response to a delta-function acceleration. While the normalized inviscid-unsteady force depends only on τ, the normalized viscous-unsteady force also depends on both Kn' and μ_b/μ through $C(\tau)$. The above force response to a delta-function acceleration can be used to define inviscid- and viscous-unsteady force kernels as

$$K_{\mathrm{iu}}(\tau) = e^{-\tau}\cos\tau,$$
$$K_{\mathrm{vu}}(t) = \frac{C(c_0 t/a)}{\sqrt{t}} = C(c_0 t/a) K_B(t), \tag{19}$$

where $K_B(t)$ is the Basset history kernel.

We have carried out numerical simulations for $\mu_b = 0$, wherein the spherical particle is initially stationary in a quiescent fluid and impulsively accelerated to a final steady state. To extract the unsteady force, we subtract the quasisteady standard drag. The results of the simulations are shown in Fig. 2, where the nondimensional unsteady force is plotted as a function of the acoustic time $\tau = c_0 t/a$. The agreement between the theory and the simulations is excellent. The simulations capture accurately the $\tau^{-1/2}$ decay for both $\tau \ll 1$ and $\tau \gtrsim O(10)$. At intermediate times, the influence of the inviscid-unsteady force becomes significant.

In the first three simulations, we have $1/(\mathrm{ReM}) = \{10^4, 10^3, 10^2\}$, respectively, and thus the agreement between the nonlinear simulations and the linear theory is good over the entire range of the computed time interval.

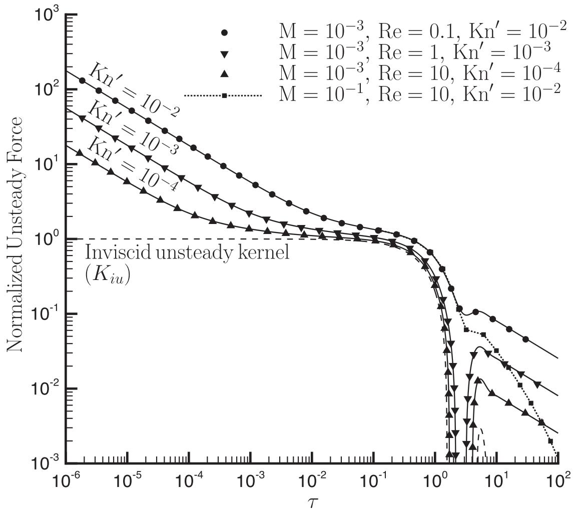

FIG. 2. Time evolution of the normalized unsteady force. Theoretical predictions [last two terms of Eq. (18)] are plotted as solid lines for $\mu_b = 0$. Corresponding simulation results for four different cases are shown as symbols.

We have also simulated a case of $\mathrm{M} = 10^{-1}$ and $\mathrm{Re} = 10$, corresponding to $\mathrm{Kn}' = 10^{-2}$. Since $1/(\mathrm{ReM}) = 1$ for this case, the effect of nonlinearity becomes important for $\tau \approx O(1)$ and the results of the simulation show a faster decay than $\tau^{-1/2}$.

As $\tau \to 0$, while the inviscid kernel is equal to unity, the viscous kernel diverges as $1/\sqrt{\tau}$. At large times, the inviscid kernel decays exponentially, while the viscous kernel decays algebraically. Thus, both at short and long times, the viscous-unsteady force dominates the inviscid-unsteady force. It can be shown that for $\mathrm{Kn}' < 5.96 \times 10^{-2}$ there exists an intermediate range of time where the inviscid-unsteady force will exceed the viscous-unsteady force.

Generalization of the BBO equation to compressible flows.—The above-presented results can be used to write a generalization of the BBO equation to compressible flow as

$$m_p \frac{d\mathbf{v}}{dt} = -6\pi a \mu \mathbf{v} - m_f \int_{-\infty}^{t} K_{\mathrm{iu}}\left((t-\xi)\frac{c_0}{a}\right) \times \frac{d\mathbf{v}}{dt}\Big|_{t=\xi} d\left(\xi \frac{c_0}{a}\right) - 6a^2 \rho_0 \sqrt{\pi\nu} \int_{-\infty}^{t} K_{\mathrm{vu}}(t-\xi) \times \frac{d\mathbf{v}}{d\xi}\Big|_{t=\xi} d\xi, \tag{20}$$

where the inviscid- and viscous-unsteady force kernels are given in Eq. (19). The significance of this extension is twofold. First, it includes explicit expressions for the inviscid-unsteady and viscous-unsteady components of the force that reduce to their well-known counterparts in the incompressible limit. Second, the extension can be combined with other forces such as buoyancy and lift to give a complete equation of motion.

When the proposed equation of motion (20) is used in practice, an expression for $C(\tau)$ is required. A curve fit with less than 1% error (see Fig. 1) has been given in Parmar [14] with more detailed discussion.

Finally, it should also be pointed out that the kernels presented in Eq. (19) combined with the correction function $C(\tau)$ are appropriate in the limit of $\mathrm{Re} \to 0$ and $\mathrm{M} \to 0$. The finite M influence on the inviscid kernel has been addressed by Parmar *et al.* [12]. Similarly, the correction function $C(\tau)$ can be expected to depend on both Re and M.

Conclusions.—We have obtained an explicit equation for the time-dependent force on a spherical particle undergoing arbitrary unsteady motion in a compressible flow. The resulting equation of motion is the generalization of the Basset-Boussinesq-Oseen equation to the compressible regime. The significance of this extension is that it includes explicit expressions for the quasisteady, inviscid-unsteady, and viscous-unsteady components of the force, which reduce to their well-known counterparts in the incompressible limit. The effect of compressibility on the inviscid-unsteady force is significant, while the effect on the viscous-unsteady force is modest. The modification due to compressibility appears as a multiplicative correction function $C(c_0 t/a)$ to the Basset history force, whose value is bounded, $4/9 \leq C(c_0 t/a) < 1.5$ (for zero bulk viscosity). The effect of compressibility on the inviscid-unsteady and the viscous-unsteady force is significant only up to few acoustic times, say $t < 10a/c_0$.

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