Spectral Slope and Kolmogorov Constant of MHD Turbulence

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The spectral slope of strong MHD turbulence has recently been a matter of controversy. While the Goldreich-Sridhar model predicts a $-5/3$ slope, shallower slopes have been observed in numerics. We argue that earlier numerics were affected by driving due to a diffuse locality of energy transfer. Our highest-resolution simulation ($3072^2 \times 1024$) exhibited the asymptotic $-5/3$ scaling. We also discover that the dynamic alignment, proposed in models with $-3/2$ slope, saturates and cannot modify the asymptotic, high Reynolds number slope. From the observed $-5/3$ scaling we measure the Kolmogorov constant $C_{\rm KA} = 3.27 \pm 0.07$ for Alfvénic turbulence and $C_K = 4.2 \pm 0.2$ for full MHD turbulence, which is higher than the hydrodynamic value of 1.64. This larger C_K indicates inefficient energy transfer in MHD turbulence, which is in agreement with diffuse locality.

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Introduction.—The equations of incompressible ideal magnetohydrodynamics, written in terms of Elsasser variables closely resemble Euler's equation.

$$\partial_t \mathbf{w}^\pm + \hat{S}(\mathbf{w}^\mp \cdot \nabla)\mathbf{w}^\pm = 0, \quad (1)$$

where $\mathbf{w}^\pm = \mathbf{v} \pm \mathbf{b}$, $\mathbf{b} = \mathbf{B}/(4\pi\rho)^{1/2}$, $\hat{S} = (1 - \nabla\Delta^{-1}\nabla)$. This resemblance is misleading, as the local mean-field $\mathbf{b}$, known as Alfvén speed $\mathbf{v}_A$, will dominate the dynamics on small scales [1]. A proper perturbation theory, however, revealed that MHD turbulence has a tendency of becoming *stronger* on smaller scales [2]. The turbulence strength, approximated as $\xi = wk_\perp/v_A k_\|$, which is the ratio of the mean-field term to the nonlinear term, will increase due to the increasing anisotropy $k_\perp/k_\|$. As turbulence becomes marginally strong ($\xi \sim 1$), the cascading time scales become close to the dynamical time scales $\tau_{\rm casc} \sim \tau_{\rm dyn} = 1/wk_\perp$. The perturbation frequency ω, however, has a lower bound due to an uncertainty relation $\tau_{\rm casc}\omega > 1$ [3]. Another bound on ω due to the *directional* uncertainty of the $\mathbf{v}_A$ is consistent with previous bound in case of "balanced" turbulence with $\delta w^+ \approx \delta w^-$, which we consider in this Letter, but lead to a modified relation in the "imbalanced" case [4]. The combination of the lower bound $\xi \leq 1$ and the tendency of nonlinear interactions to increase ξ will lock ξ "critically balanced" with $\xi \sim 1$. Thus the cascade will be strong and will have a Kolmogorov's $-5/3$ spectrum. The perturbations will be anisotropic with respect to the local mean magnetic field with $k_\| \sim k_\perp^{2/3}$ [3].

One can further simplify Eq. (1) by neglecting the term $(\delta w_\|^\mp \nabla_\|)\delta w^\pm$ which is much smaller than the mean-field term $(v_A \nabla_\|)\delta w^\pm$. After this Eq. (1) splits into two equations, one for $\delta w_\|^\pm$, which, in this strongly anisotropic case, $k_\| \ll k_\perp$ represents slow (or pseudo-Alfvén) mode and the equation for $\delta w_\perp^\pm$ which represent Alfvénic mode. The equation for slow mode is passive and does not provide any back-reaction for the Alfvénic equation $\partial_t \delta\mathbf{w}_\perp^\pm \mp (\mathbf{v_A} \cdot \nabla_\|)\delta\mathbf{w}_\perp^\pm + \hat{S}(\delta\mathbf{w}_\perp^\mp \cdot \nabla_\perp)\delta\mathbf{w}_\perp^\pm = 0$, known as reduced MHD approximation or RMHD [5]. RMHD has a precise two-parametric symmetry with respect to the anisotropy and the strength of the mean-field: $\mathbf{w} \to \mathbf{w}A$, $\lambda \to \lambda B$, $t \to tB/A$, $\Lambda \to \Lambda B/A$, which is similar to hydrodynamic symmetry. Here, λ is a perpendicular scale, Λ is a parallel scale, A and B are arbitrary parameters. Because of this precise symmetry one can hypothesize that strong Alfvénic turbulence has a universal regime, similar to hydrodynamic universal cascade of Kolmogorov [6]. In nature, the universal regime for MHD can be achieved with $\delta w^\pm \ll v_A$. In numerical simulations, we can directly solve RMHD equations, which have precise symmetry already built in. From practical viewpoint, the statistics from the full MHD simulation with $\delta w^\pm \sim 0.1 v_A$ is virtually indistinguishable from RMHD statistics and even $\delta w^\pm \sim v_A$ are fairly similar to the former [7]. In this Letter we use both full MHD simulations and RMHD simulations. Statistically isotropic MHD simulation is used to determine a fraction of total energy contained in the slow mode, while RMHD simulations are used to study properties of the universal Alfvénic turbulence. Previous numerical work confirmed scale-dependent anisotropy of the strong MHD turbulence [8]. The precise value of the spectral slope, however, was a matter of debate. In particular, [9] claimed that the mean-field strong turbulence has a slope of $-3/2$. This motivated adjustments to the Goldreich-Sridhar model [10–12]. A model with so-called "dynamic alignment" [11,13] became popular after the scale-dependent alignment was discovered in numerical simulations [14]. This model is based on the idea that the alignment between velocity and magnetic perturbations decreases the strength of the interaction, also it assumes that the alignment is a power-law function of scale, increasing indefinitely towards small scales and modifying the spectral slope of MHD turbulence from the $-5/3$ slope

TABLE I. Three-dimensional simulations.

Run	Type	$n_x n_y n_z$	Dissipation	$\langle\epsilon\rangle$	L/η
H1	hydro	512^3	$3.02 \times 10^{-4} k^2$	0.091	190
H2	hydro	1024^3	$1.20 \times 10^{-4} k^2$	0.091	370
M1	MHD	1024^3	$1.63 \times 10^{-9} k^4$	0.159	280
R1	RMHD	256×768^2	$6.82 \times 10^{-14} k^6$	0.073	280
R2	RMHD	512×1536^2	$1.51 \times 10^{-15} k^6$	0.073	570
R2.5	RMHD	1536^3	$1.51 \times 10^{-15} k^6$	0.073	570
R3	RMHD	1024×3072^2	$3.33 \times 10^{-17} k^6$	0.073	1100

to $-3/2$ slope. In this Letter we critically examine the assumption that the alignment is a power-law function of scale. We also show that earlier measurements of the MHD slope were premature due to diffuse scale-locality of the energy transfer in MHD turbulence.

Numerical methods.—Pseudospectral dealiased code was used to solve hydrodynamic, MHD and RMHD equations. The right-hand side of Eq. (1) had an explicit dissipation term $-\nu_n(-\nabla^2)^{n/2}\mathbf{w}^\pm$ and forcing term $\mathbf{f}$. The code and the choice for numerical resolution, driving, etc, was described in great detail in our earlier publications [7,15,16]. Table I shows the parameters of the simulations. The Kolmogorov scale is defined as $\eta = (\nu_n^3/\epsilon)^{1/(3n-2)}$, the integral scale $L = 3\pi/4E \int_0^\infty k^{-1}E(k)dk$ (which was approximately 0.79 for R1-3). Dimensionless ratio L/η could serve as a "length of the spectrum", although usually spectrum is around an order of magnitude shorter.

The resolution in the direction parallel to the mean magnetic field, n_x was reduced by a factor of 3 for simulations R1-3. This was possible due to an empirically known lack of energy in the parallel direction in k space. We ran a simulation R2.5 which has full resolution in n_x to compare with R2 and check the influence of this resolution reduction on the power spectrum. Although the bottleneck effect was slightly less pronounced in R2.5 compared to R2, there was only a small influence in the inertial range. We conclude that using n_x reduced by a factor of 2 or 3 is possible.

For the purpose of this Letter we used driving that had a constant energy injection rate. In RMHD simulations R1-3 we drove turbulence to the amplitude that it will be strong on the outer scale. R1-3 were started from lower-resolution simulation that reached stationary state and were further evolved in high resolution for approximately 12 Alfvénic times, which, for strong MHD turbulence also correspond to about 12 dynamical times. The averaged quantities were obtained for the last 6 Alfvénic times. In all magnetic simulations M1, R1-3, we were using hyperviscosity ($n > 2$) instead of normal viscosity. This is possible due to the fact that bottleneck effect is much less pronounced in the MHD case, compared to hydro.

Spectra and universality.—Much of the study of hydrodynamic turbulence was dedicated to Kolmogorov model which assumes a universal cascade of energy through scales [6]. This model predicts that the power spectrum of velocity, $E(k)$, will be a power-law function of wave number k, $E(k) = C_K\epsilon^{2/3}k^{-5/3}$, where C_K is called Kolmogorov constant. This scaling has an intermittency correction $(kL)^\alpha$, where L is an outer scale and $\alpha \approx 0.035$ [17], but in simulations or measurements with small inertial range it can neglected. A compilation of experimental results for hydrodynamic turbulence [18] suggests that a Kolmogorov constant is universal for a wide variety of flows. High-resolution simulations of isotropic incompressible hydrodynamic turbulence [19] suggest the same value for this universal constant.

A robust method for determining the spectral slope and the Kolmogorov constant from simulations is a resolution study [19], when a number of numerical experiments are performed with different resolution and the spectra are plotted with respect to the dimensionless wavevector, $k\eta$. A physical meaning of such a comparison is based on an assumption that a simulation with higher numerical resolution can be considered both as a simulation resolving smaller physical scales and as a simulation of a larger volume of turbulence, see Fig. 1. This assumption is true as long as turbulence can be considered scale-local, i.e., the effects of driving can be neglected in the inertial range. Our hydrodynamic simulations reveal a good convergence of spectra with numerical resolution and show a universal Kolmogorov constant consistent with the one obtained in [19]. Also the shape of the dissipation range is similar to the one in aforementioned paper. Despite moderate resolution, the inertial ranges converge, which is due to locality of hydrodynamic cascade in spacial scales, making it possible to consider higher and lower-resolution simulations on common ground, neglecting the influence of large scales, where energy is provided by driving.

Figure 2 presents a resolution study for simulations R1-3 determining the spectral slope and Kolmogorov constant for Alfvénic turbulence. First, we note that if MHD turbulence had the above mentioned spectral slope of $-3/2$ the outer scale point, marked by a cross on Fig. 2 would be going down between R1 to R3 by a factor of 1.26. Instead, it stays at about the same level, indicating that deviations

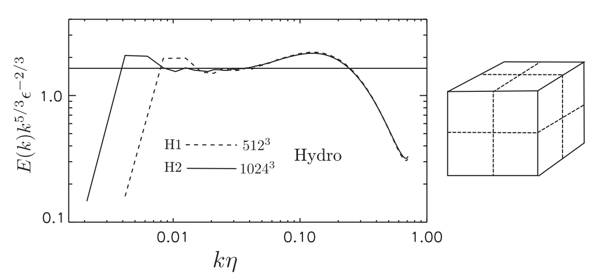

FIG. 1. Our hydrodynamic simulations H1 and H2 reproduce well $C_K = 1.64$ reported in earlier studies [19]. The cube on the right illustrates that the small scale dynamics in large 1024^3 simulation reproduces small scale dynamics in eight 512^3 simulations as long as turbulence is scale-local and the effects of large scales could be neglected in the inertial range. This is confirmed by numerical convergence of H1 and H2.

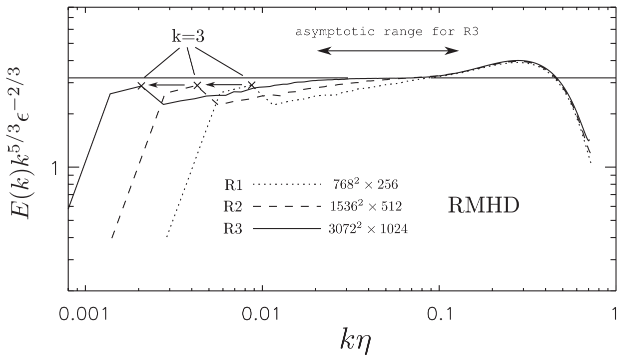

FIG. 2. Resolution study for RMHD simulations R1-R3. Both axis are dimensionless. The numerical convergence is observed at the dissipation scale, which features well-defined bottleneck bump and at the inertial range which is immediately adjacent to the dissipation scale. The scales within the order of magnitude of the driving scale do not show convergence. This is an indication that MHD turbulence is less local than hydrodynamic turbulence, in agreement with an earlier claim [15]. Thus, convergence in MHD requires higher-resolution simulations.

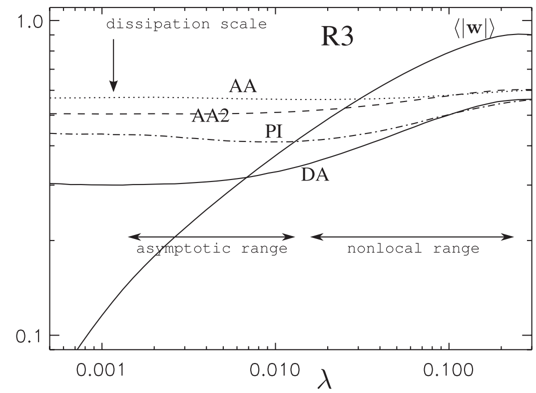

FIG. 3. Dynamic alignment saturates towards small scales *before* the dissipation scale. This indicates that the scale-dependent alignment is a transient effect that is present in simulations due to MHD turbulence being less local. The asymptotic regime of MHD turbulence is showing constant alignment, which will not modify the spectral slope $-5/3$ of strong MHD turbulence (GS95 slope).

from $-5/3$ Kolmogorov slope are empirically known to be small. The flat part of the normalized spectrum on Fig. 2 in R3 simulation between $k = 54(k\eta \approx 0.037)$ and $k = 91(k\eta \approx 0.063)$ with central frequency $k = 70$ was fit to obtain Kolmogorov constant. The value obtained in this fit was $C_K = 3.27 \pm 0.07$ where the error was mostly due to fluctuations of normalized spectrum with time.

Dynamic alignment.—It was suggested in [11,13] that the spectral slope of MHD turbulence is modified by so-called "dynamic alignment" that increases indefinitely towards small scales as $\lambda^{1/4}$. In our earlier studies [14,15] we measured several types of alignment and found no evidence that all alignment measures follow the same scaling. In this Letter we confirm this finding with higher-resolution simulations, in addition we found that all alignment measures saturate, i.e., approach an asymptotic constant value on small scales. Figure 3 shows the alignment measures in R3, where AA, AA2, DA, and PI are different alignment measures: $AA = \langle|\delta\mathbf{w}_\lambda^+ \times \delta\mathbf{w}_\lambda^-|/|\delta\mathbf{w}_\lambda^+||\delta\mathbf{w}_\lambda^-|\rangle$, $AA2 = \langle|\delta\mathbf{v}_\lambda^+ \times \delta\mathbf{b}_\lambda^-|/|\delta\mathbf{v}_\lambda^+||\delta\mathbf{b}_\lambda^-|\rangle$, $PI = \langle|\delta\mathbf{w}_\lambda^+ \times \delta\mathbf{w}_\lambda^-|\rangle/\langle|\delta\mathbf{w}_\lambda^+||\delta\mathbf{w}_\lambda^-|\rangle$, $DA = \langle|\delta\mathbf{v}_\lambda^+ \times \delta\mathbf{b}_\lambda^-|\rangle/\langle|\delta\mathbf{v}_\lambda^+||\delta\mathbf{b}_\lambda^-|\rangle$, for motivation and details see [14,15]. The ratio of smallest values of DA in R2 and R3 was 1.02 which is inconsistent with $2^{1/4} \approx 1.19$ predicted by [11,13].

We are not aware of any convincing physical argumentation explaining why alignment should be a power-law of scale. Reference [13] argues that alignment will tend to increase, but will be bounded by field wandering; i.e., the alignment on each scale will be created independently of other scales and will be proportional to the relative perturbation amplitude $\delta B/B$. But this violates two-parametric symmetry of RMHD equations mentioned above, which suggests that field wandering cannot destroy alignment or imbalance. Indeed, a perfectly aligned state, e.g., with $\delta\mathbf{w}^- = 0$ is a precise solution of MHD equations and it is not destroyed by its own field wandering. The alignment measured in simulations of strong MHD turbulence with different values of $\delta B_L/B_0$ showed very little or no dependence on this parameter [15]. Figure 3 also demonstrates a difference in scaling between alignment measures and the first-order structure function of the amplitude, i.e., $\langle|\delta\mathbf{w}_\lambda|\rangle$. According to [13] they should scale the same way, but this is not observed. The transition region $\lambda = 0.02$–0.2 for alignment is so wide, due to the diffuse locality [15].

To summarize, our numerical data are consistent with alignment measures becoming constant in the inertial range and inconsistent with the hypothesis that they depend as $\lambda^{1/4}$ on scale. This crucial observation suggests that the alignment of the fields is unlikely to significantly modify the expected $-5/3$ spectral slope of strong MHD turbulence with mean field, proposed in [3], also we do not see any reason to favor the $-3/2$ slope.

The amount of slow mode and the total Kolmogorov constant for MHD turbulence.—Full incompressible MHD turbulence have a cascade of slow mode, which was not included in our RMHD simulations R1-3. Although in nature slow mode is often damped, it is normally present in full MHD incompressible simulations, e.g., the ones presented in [9,20]. The passive cascade of slow mode will have the same energy spectrum as the Alfvénic mode, and the *total* Kolmogorov constant for MHD turbulence will be expressed as

$$C_K = C_{\mathrm{KA}}(1 + C_s)^{1/3}, \qquad (2)$$

where C_s is the ratio of slow to Alfvénic energies. This ratio could depend on how MHD turbulence is driven. Many previous studies simulated MHD turbulence with zero mean field and statistically isotropic forcing, which is motivated by the lack of imposed field in astrophysical systems. These studies estimated Kolmogorov constant directly [20]. Our study uses a less straightforward approach, measuring C_s from a simulation with zero mean field and substituting it into Eq. (2). This approach is motivated by our finding that MHD turbulence is less local and therefore it is much harder to achieve an asymptotic universal cascade using zero-mean-field simulation. The transition to the strong local mean-field case will require at least a couple of orders of magnitude in scale and the subsequent transition to universal cascade, as we observed in the previous section, will take about 2 orders of magnitude in scale. Therefore, in currently available zero mean-field simulations it is impossible to observe the universal cascade.

The "natural" value of C_s is unity, because the incompressible MHD equations have 4 degrees of freedom, out of which Alfvénic mode uses two and slow mode also uses two. Having the same amount of degrees of freedom and the isotropic driving that does not prefer any direction we would expect that the energy will be distributed equally between the modes. We measured how energy is partitioned on small scales of simulation M1 by making a local Fourier transform of smaller cubes and decomposing into modes with respect to the local mean field. The measured partition of energy is $C_s \approx 1.3$. Although statistical errors in this measurement are small, in a simulation with finite resolution C_s could deviate from its asymptotic value. Conservatively, we will assume that C_s is between 1, which is equipartition, and 1.3, which is observed in our simulation M1. The total Kolmogorov constant, therefore, will be estimated as 4.2 ± 0.2.

Scale locality and Kolmogorov constant.—The contribution to energy flux through a particular scale from different k wavebands have an analytical upper bound that can be interpreted as a scale-locality constraint [21]. This upper bound, however, is *absolute*, while the locality from practical viewpoint is constrained by a measure *relative* to the actual energy flux. Therefore, this locality constraint depends on the efficiency of the energy transfer, such as the efficient energy transfer must be local, while inefficient one could be nonlocal [16]. Quantitatively, the upper bound on the ratio of scales significantly contributing to the energy transfer will scale asymptotically as $C_K^{9/4}$. As we observe larger C_K in MHD turbulence compared to hydrodynamic turbulence, the former could be less local than the latter, which is consistent with our earlier findings [15].

Discussion.—Previous measurements of the slope usually relied on the highest-resolution simulation and fitted the slope in the fixed k range close to driving scale typically between $k = 5$ and $k = 20$. In this Letter we argue that such a fit is unphysical and instead one should fit a fixed $k\eta$ range. In the former case the result would be a shallower spectral slope due to proximity to the outer scale and driving. In the latter case the effect of the driving will diminish with increasing resolution and one will observe shallower spectra at small resolutions that will become steeper with increasing resolution. For a more general imbalanced case and transition to the balanced limit considered in this Letter see [4,7,16] and references therein.

Earlier measurements of Kolmogorov constant in MHD turbulence reported lower values than this study, e.g., $C_K = 2.2$ in [20]. We believe this is due to insufficient resolution in those simulations, which prevented the observation of the asymptotic regime. In particular, in the case of statistically isotropic simulations like the ones in [20] a transition to small scale sub-Alfvénic regime precede the transition to asymptotic regime. These two transitions require numerical resolution that is even higher than the highest resolution presented in this Letter and for now seems computationally impossible. Our own statistically isotropic simulation M1 shows Kolmogorov constant of 3.5, which is still only a lower limit, consistent with 4.2 ± 0.2 derived here.

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[1] P.S Iroshnikov, Sov. Astron. **7**, 566 (1963); R.H. Kraichnan, Phys. Fluids **8**, 1385 (1965).
[2] S. Galtier *et al.*, J. Plasma Phys. **63**, 447 (2000).
[3] P. Goldreich and S. Sridhar, Astrophys. J. **438**, 763 (1995).
[4] A. Beresnyak and A. Lazarian, Astrophys. J. **682**, 1070 (2008).
[5] B.B. Kadomtsev and O.P. Pogutse, Sov. Phys. JETP **38**, 283 (1974); H.R. Strauss, Phys. Fluids **19**, 134 (1976).
[6] A. Kolmogorov, Dokl. Akad. Nauk SSSR **31**, 538 (1941).
[7] A. Beresnyak and A. Lazarian, Astrophys. J. **702**, 460 (2009).
[8] J. Cho and E. Vishniac, Astrophys. J. **539**, 273 (2000); J. Maron and P. Goldreich, Astrophys. J. **554**, 1175 (2001).
[9] W.-C. Müller and R. Grappin, Phys. Rev. Lett. **95**, 114502 (2005).
[10] S. Galtier, A. Pouquet, and A. Mangeney, Phys. Plasmas **12**, 092310 (2005).
[11] S. Boldyrev, Astrophys. J. **626**, L37 (2005).
[12] G. Gogoberidze, Phys. Plasmas **14**, 022304 (2007).
[13] S. Boldyrev, Phys. Rev. Lett. **96**, 115002 (2006).
[14] A. Beresnyak and A. Lazarian, Astrophys. J. Lett. **640**, L175 (2006).
[15] A. Beresnyak and A. Lazarian, Astrophys. J. **702**, 1190 (2009).
[16] A. Beresnyak and A, Lazarian, Astrophys. J. **722**, L110 (2010).
[17] Z. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994); Y. Kaneda *et al.*, Phys. Fluids **15**, L21 (2003).
[18] K.R. Sreenivasan, Phys. Fluids **7**, 2778 (1995).
[19] P.K. Yeung and Y. Zhou, Phys. Rev. E **56**, 1746 (1997); T. Gotoh, D. Fukayama, and T. Nakano, Phys. Fluids **14**, 1065 (2002).
[20] D. Biskamp, *Magnetohydrodynamic Turbulence* (Cambridge University Press, Cambridge, England, 2003).
[21] H. Aluie and G.L. Eyink, Phys. Rev. Lett. **104**, 081101 (2010).