Spatially Modulated Phase in the Holographic Description of Quark-Gluon Plasma

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We present a string theory construction of a gravity dual of a spatially modulated phase. Our earlier work shows that the Chern-Simons term in the five-dimensional Maxwell theory destabilizes the Reissner-Nordström black holes in anti–de Sitter space if the Chern-Simons coupling is sufficiently high. In this Letter, we show that a similar instability is realized on the world volume of 8-branes in the Sakai-Sugimoto model in the quark-gluon plasma phase. Our result suggests a new spatially modulated phase in quark-gluon plasma when the baryon density is above $0.8N_f$ fm^{-3} at temperature 150 MeV.

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Introduction.—Five-dimensional Maxwell theory with a Chern-Simons term is tachyonic in the presence of a constant electric field [1]. The tachyonic modes with nonzero spatial momenta destabilize the Reissner-Nordström black holes in five-dimensional anti–de Sitter space (AdS_5) if the Chern-Simons coupling is larger than a certain critical value. If its holographically dual quantum field theory exists, the instability would imply a spatially modulated phase transition in the theory. It would be interesting to construct a model exhibiting such an instability *ab initio* from superstring theory so that we can be certain that the dual theory exists and know what it is. This has not been demonstrated so far. For example, it was shown in [1] that the three-charge extremal black hole in the type IIB superstring theory on $AdS_5 \times S^5$ is barely stable.

In this Letter, we show that such an instability is realized in the quark-gluon plasma phase of the Sakai-Sugimoto model for QCD with N_f flavors of massless quarks [2]. On the world volume of the 8-branes, there is a $U(N_f)$ gauge field, and its diagonal $U(1)$ part is dual to the quark number ($= N_c$ times the baryon number). The baryons are identified with instanton solutions on the world volume in this model [3]. World volume solutions representing QCD states with finite baryon density and temperature have been studied [2,4–8].

Most of the solutions with finite baryon density are singular at the sources of baryons charges, and it is not clear whether the supergravity approximation is applicable. One of the exceptional cases is the quark-gluon plasma phase, where there is a smooth solution representing a finite baryon density configuration.

In the Sakai-Sugimoto model, the gluon degrees of freedom are realized on N_c D4-branes compactified on a circle S^1_c with supersymmetry breaking boundary condition [9]. At finite temperature, we compactify the Euclidean time on another circle S^1_T, and the D4-brane world volume has the topology of $S^1_c \times S^1_T \times \mathbb{R}^3$. In the confinement phase, S^1_c is contractible in the bulk, and the topology of the bulk geometry is then $S^1_T \times \mathbb{R}^3 \times S^4$ times a disk bounded by S^1_c. Each 8-brane wraps the thermal $S^1_T \times S^4$ and is extended in $\mathbb{R}^3$. In this phase, the 8-brane starts as a D8-brane at a point on S^1_c, meanders in the bulk, and ends as a $\overline{\mathrm{D8}}$-brane at another point on S^1_c.

In the deconfinement phase, the thermal S^1_T becomes contractible in the bulk geometry [9]. Depending on the relative locations of the 8-branes, the chiral symmetry restoration takes place at or above the deconfinement temperature [2,4,5]. Above the chiral symmetry restoration temperature, D8 and $\overline{\mathrm{D8}}$-branes become separated, and each of them has the topology of a disk bounded by S^1_T times S^4. This describes a holographic dual of the quark-gluon plasma. In this phase, it is possible to construct a solution with finite baryon density that is smooth everywhere on the world volume, as we will discuss below. In this Letter, we will focus on this case.

The dynamics of the 8-brane world volume is described by the Dirac-Born-Infeld (DBI) action with the Chern-Simons term. We show that there is a critical baryon density above which the brane configuration becomes unstable by tachyonic modes carrying nonzero momenta. This suggests a spatially modulated phase with a baryon density wave.

A holographic dual of a baryon density wave was discussed in the "bottom-up" model in [10]. The instability of the Sakai-Sugimoto model has been studied earlier, for example, in [11], but not in the chiral symmetric phase. To our knowledge, it has not been shown whether the Chern-Simons coupling on the world volume theory on the 8-branes is large enough to trigger the spatially modulated phase transition. In this Letter, we give the first demonstration of a spatially modulated phase transition in a "top-down" model with a well-understood dual pair.

Instability of homogeneous solution.—The bulk geometry above the deconfining temperature is the near horizon geometry of the N_c D4-branes at finite temperature compactified on the supersymmetry breaking circle S^1_c [9]. In the notation of [12], the metric is

$$ds^2 = \left(\frac{U}{R}\right)^{3/2}(-f(U)dX_0^2 + d\vec{X}^2 + dX_4^2) + \left(\frac{R}{U}\right)^{3/2}\left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \quad (1)$$

where $U_T = (\frac{4\pi}{3}RT)^2 R$ is the location of the horizon at temperature T, $f(U) = 1 - U_T^3/U^3$, $R^3 = \pi g_s N_c l_s^3$, and $d\Omega_4^2$ is a metric for a unit four-sphere. The coordinate U and the four-sphere represent the transverse directions to the D4-branes. The temperature T sets the periodicity of the imaginary time ($\mathrm{Im}X_0$) direction, while the period of the compact X_4 direction is arbitrary. In the chiral symmetry restoration phase, each 8-brane is located at a constant X_4 [2,4,5].

The D8 and $\overline{\mathrm{D8}}$-branes are separated in the chiral symmetric phase. Let us focus on the dynamics on the D8-branes. The DBI action on the D8-brane is given by

$$S = -T_{\mathrm{D8}}\int d^9\sigma e^{-\phi}\sqrt{-\det(g_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} + S_{\mathrm{CS}}, \quad (2)$$

where $T_{\mathrm{D8}} = (2\pi)^{-8} l_s^{-9}$ and the dilaton is given by $e^{\phi} = g_s(U/R)^{3/4}$. The Chern-Simons action is

$$S_{\mathrm{CS}} = \frac{1}{48\pi^3}\int_{\mathrm{D8}} F_4 \wedge \omega_5(A), \quad (3)$$

where $F_4 = dC_3$ is the RR 4-form field which satisfies $\frac{1}{2\pi}\int_{S^4} F_4 = N_c$ and $\omega_5(A)$ is the Chern-Simons 5-form.

For our purpose, it is sufficient to turn on the $U(1)$ part of the gauge field on the world volume. To the quadratic order, the $U(1)$ part does not couple to the $SU(N_f)$ part of the gauge field or fluctuations of the 8-brane in the transverse direction. Couplings to the bulk degrees of freedom are suppressed by $1/N_c$. To simplify our equations, we rescale the gauge field and the metric as $A = \frac{R^2}{2\pi\alpha'}\tilde{A}$ and $g_{\alpha\beta} = R^2\tilde{g}_{\alpha\beta}$. We also rescale the coordinates as $U = Ru$, $X_0 = Rt$, $\vec{X} = R\vec{x}$, and $X_4 = R\tau$. Following [2], we assume that the gauge field is constant on the S^4 and obtain the effective five-dimensional action,

$$S/c = -\int_{M_4\times\mathbb{R}} dt d^3x du u^{1/4}\sqrt{-\det(\tilde{g}_{\alpha\beta} + \tilde{F}_{\alpha\beta})} + \alpha\int_{M_4\times\mathbb{R}} dt d^3x du \epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5}\tilde{A}_{\mu_1}\tilde{F}_{\mu_2\mu_3}\tilde{F}_{\mu_4\mu_5}, \quad (4)$$

with the five-dimensional metric,

$$ds^2 = u^{3/2}(-f(u)dt^2 + d\vec{x}^2 + d\tau^2) + \frac{1}{u^{3/2}f(u)}du^2,$$
$$f(u) = 1 - \frac{u_T^3}{u^3}, \qquad u_T = \left(\frac{4\pi}{3}RT\right)^2. \quad (5)$$

The Chern-Simons term induces anomaly in the vector current on the boundary. In general, one needs to add the Bardeen counterterm to restore the current conservation. With only the vector electric field turned on, the counterterm vanishes. In particular, the definition of the chemical potential below is not modified by this. The Chern-Simons coupling α is fixed to be $3/4$ and the factor c is

$$c = \frac{8\pi^2}{3}T_{\mathrm{D8}}N_f g_s^{-1}R^9. \quad (6)$$

Note that, modulo the overall factor c, the action (4) depends only on u_T.

If the kinetic term for the gauge field were of the Maxwell form $\tilde{F}^2$, the electric field strength could be made arbitrarily high by raising the baryon density, and any nonzero value of the Chern-Simons coupling would induce an instability of the type discovered in [1]. With the DBI action, there is an upper bound for the field strength, and it requires a more careful analysis to determine whether the instability takes place.

Let us consider a background configuration with nonzero $\tilde{A}_0 = \tilde{A}_0(u)$. The equation of motion gives

$$\tilde{E}(u) = \frac{\tilde{\rho}}{\sqrt{\tilde{\rho}^2 + u^5}}, \quad (7)$$

where $\tilde{E} = -\tilde{F}_{tu} = \partial_u\tilde{A}_0$. The integration constant $\tilde{\rho}$ will be identified as a rescaled value of the quark density ρ ($= N_c$ times the baryon density). As advertised in the introduction, this finite quark density solution is regular everywhere on the brane. We choose the gauge so that $\tilde{A}_0(u)$ vanishes on the horizon. Note that, although the action includes the Chern-Simons term, the equations of motion are gauge invariant. The chemical potential $\tilde{\mu}$ is given by the asymptotic value of $\tilde{A}_0$ at $u \to \infty$.

$$\tilde{\mu} = \tilde{A}_0(u = \infty) = \int_{u_T}^{\infty} du \frac{\tilde{\rho}}{\sqrt{\tilde{\rho}^2 + u^5}}. \quad (8)$$

Let us perturb this configuration as $\tilde{F} \to \tilde{F} + \delta\tilde{F}$. To find an onset of the instability, we look for a static normalizable solution in the linearized equation for $\delta\tilde{F}$,

$$\partial_u\left[\frac{u^{5/2}f(u)}{\sqrt{1-\tilde{E}(u)^2}}\delta\tilde{F}_{ui}\right] - u^{-1/2}\sqrt{1-\tilde{E}(u)^2}\partial_j\delta\tilde{F}_{ij} - 2\alpha\epsilon_{ijk}\tilde{E}(u)\delta\tilde{F}_{jk} = 0. \quad (9)$$

If we apply the operator $\epsilon_{ijk}\partial_j$ and use the Fourier mode $\delta\tilde{F}_{ij} = \epsilon_{ijk}v_k e^{-ik_l x^l}\phi(u)$ with an eigenvalue $ik = i|\vec{k}|$ (the eigenvalue $-ik$ gives the same result), $\phi(u)$ obeys a second order ordinary differential equation,

$$\left[-\frac{d}{du}f(u)\sqrt{\tilde{\rho}^2 + u^5}\frac{d}{du} + \frac{-4\alpha\tilde{\rho}k + u^2k^2}{\sqrt{\tilde{\rho}^2 + u^5}}\right]\phi(u) = 0. \quad (10)$$

At the horizon $u = u_T$, we use the ingoing boundary condition for static waves.

We solved the linearized Eq. (10) numerically for general values of the Chern-Simons coupling α. For each value of the Chern-Simons coupling $\alpha > 1/4$, we found a critical value of $\tilde{\rho}$ above which the instability takes place. Figure 1 depicts the critical quark density $\tilde{\rho}_{\rm crit}$ as a function of α. We note that $\tilde{\rho}_{\rm crit}$ diverges as $\alpha \to 1/4$.

We can also show analytically that $\alpha = 1/4$ is the limiting value of the Chern-Simons coupling. Let us rescale variables as $\bar{u} = \tilde{\rho}^{-(2/5)}u$, $\bar{k} = \tilde{\rho}^{-(1/5)}k$, and take the limit $\tilde{\rho} \to \infty$ in Eq. (10). We find

$$\left[-\frac{d}{d\bar{u}}\sqrt{1+\bar{u}^5}\frac{d}{d\bar{u}} + \frac{-4\alpha\bar{k} + \bar{u}^2\bar{k}^2}{\sqrt{1+\bar{u}^5}}\right]\tilde{\phi}(\bar{u}) = 0. \quad (11)$$

We have verified that a solution to this equation approaches the solution to (10) in the sense of the $\mathcal{L}^2$ measure. From the numerical evaluation of (11), we find that the momentum $\bar{k}$ with nontrivial normalizable solutions tends to infinity as we take $\tilde{\rho} \to \infty$ and α approaches the limiting value. Anticipating this, we take $\bar{k} \to \infty$ in (11) while keeping $v = \sqrt{\bar{k}}\bar{u}$ and obtain,

$$\left(-\frac{d^2}{dv^2} - 4\alpha + v^2\right)\tilde{\phi}(\bar{u}) = 0. \quad (12)$$

This can be solved by the harmonic oscillator ground state $\tilde{\phi}(v) = e^{-v^2/2}$ with $\alpha = 1/4$.

In the quark-gluon plasma phase, the Chern-Simons coupling on the world volume theory is $\alpha = 3/4$ and is above the limiting value of $1/4$. At this value of α, the critical quark density is numerically evaluated as

$$\tilde{\rho}_{\rm crit} = 3.714 u_T^{5/2}. \quad (13)$$

Let us express the critical quark density in the original set of variables. The quark density ρ is defined by a variation of the Lagrangian density by $E = \partial_u A_0$. Note we rescaled the action by the factor c in (6) and the gauge field is rescaled as $A = \frac{R^2}{2\pi\alpha'}\tilde{A}$. We should also remember that we rescaled our spacetime coordinates by R. The physical quark density ρ is then related to $\tilde{\rho}$ above as

$$\rho = c\left(\frac{R^2}{2\pi\alpha'}\right)^{-1}\frac{\tilde{\rho}}{R^3} = \frac{2}{3(2\pi)^5}\frac{N_f}{g_s}\frac{R^4}{l_s^7}\tilde{\rho}. \quad (14)$$

Substituting (13) into this, the critical quark density at $\alpha = 3/4$ is given as

$$\rho_{\rm crit} = c_0 N_f N_c (g_s N_c l_s)^2 T^5, \quad (15)$$

where $c_0 = 3.714(2/3)^6\pi^3 \approx 10$.

It is important to make sure that we can ignore backreaction of the quark density to the bulk geometry. Note that the critical baryon density is given by dividing the quark density $\rho_{\rm crit}$ by N_c and that the result is proportional to $N_f(g_sN_c)^2T^5$. The N_c dependence comes only in the combination of the 't Hooft coupling g_sN_c, which is kept finite in the large N_c limit. Since the baryons can be thought of as D4-branes wrapping S^4 [13,14], their backreaction becomes significant only when their density scales as N_c or more and is negligible in the large N_c limit provided $N_f \ll N_c$. Another way to see this is to evaluate the energy density due to the electric field using the action (4) and show that it is proportional to N_f/g_s times some power of g_sN_c. This is the same scaling behavior as the tension of the N_f 8-branes, which does not generate backreaction.

It is an interesting exercise to express the critical density in terms of QCD quantities. The string parameters g_s and l_s are related to the Yang-Mills coupling $g_{\rm YM}$ and the Kaluza-Klein scale $M_{\rm KK}$ for the compactification circle S_c^1 as $g_{\rm YM}^2 = 4\pi^2 g_s l_s/L$ and $M_{\rm KK} = 2\pi/L$, where L is the circumference of S_c^1 [12]. The critical baryon density can then be written as

$$\frac{\rho_{\rm crit}}{N_c} = \frac{c_0 N_f}{4\pi^2}\frac{\lambda^2}{M_{KK}^2}T^5, \quad (16)$$

where $\lambda = g_{\rm YM}^2 N_c$. The constants $M_{\rm KK}$ and λ can be determined by fitting, for example, with the pion decay constant and the mass of the ρ meson, as $M_{\rm KK} = 949$ MeV and $\lambda = g_{\rm YM}^2 N_c = 16.6$ [2]. The deconfinement temperature, where the thermal cycle S_T^1 becomes contractible, is at $M_{\rm KK}/2\pi = 151$ MeV. Interestingly, this turns out to be close to the critical temperature expected for the quark-gluon plasma [15]. If we substitute $T = 150$ MeV in (16), for example, the critical baryon density comes out as

$$\frac{\rho_{\rm crit}}{N_c} \approx 0.8 N_f\ {\rm fm}^{-3}. \quad (17)$$

For $N_f = 2$, this is about 10 times the nucleon density in atomic nuclei.

At the critical density $\rho = \rho_{\rm crit}$, the instability begins to occur at the momentum $k = 2.39u_T^{1/2}$, which in the original coordinates is given by $k/R \approx 10T$. If we set $T = 150$ MeV, the momentum is about 1.5 GeV, and the corresponding wavelengh is 0.8 fm.

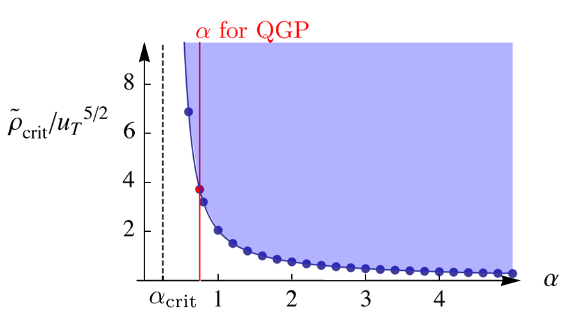

FIG. 1 (color online). The critical quark density $\tilde{\rho}$ as a function of the Chern-Simons coupling α. The instability takes place in the shaded region with $\alpha > \alpha_{\rm crit} = 1/4$. The value $\alpha = 3/4$ for the Sakai-Sugimoto model is indicated by the red vertical line.

Nonlinear solution.—We can construct a solution to the full nonlinear equations carrying a fixed nonzero momentum whose energy is lower than that of the original translationally invariant state. Following [16], we make the ansatz,

$$\tilde{A}_t = a(u), \tilde{A}_x + i\tilde{A}_y = h(u)e^{-ikz}, \tag{18}$$

with all other components vanishing. Although there may be a nonlinear solution with even lower energy, it is interesting that one can construct a candidate ground state with such a simple ansatz.

With this ansatz, the equations of motion become

$$\partial_u\left[\frac{u\sqrt{u^3+k^2h(u)^2}a'(u)}{\sqrt{1-a'(u)^2+f(u)h'(u)^2}}\right] + 4\alpha k h(u)h'(u) = 0$$
$$\partial_u\left[\frac{uf(u)\sqrt{u^3+k^2h(u)^2}h'(u)}{\sqrt{1-a'(u)^2+f(u)h'(u)^2}}\right] + 4\alpha k a'(u)h(u) - \frac{k^2uh(u)\sqrt{1-a'(u)^2+f(u)h'(u)^2}}{\sqrt{u^3+k^2h(u)^2}} = 0. \tag{19}$$

We assume that the embedding coordinate τ is constant, which is consistent with the equations of motion. The first equation can be integrated easily, and gives us the quark density,

$$\frac{u\sqrt{u^3+k^2h(u)^2}a'(u)}{\sqrt{1-a'(u)^2+f(u)h'(u)^2}} + 2\alpha k h(u)^2 = \tilde{\rho}. \tag{20}$$

Using this expression, the second equation becomes

$$K(u)\partial_u(K(u)f(u)h'(u)) - k^2u^2h(u) + 4k\alpha h(u)(\tilde{\rho} - 2k\alpha h(u)^2) = 0, \tag{21}$$

where

$$K(u) = \sqrt{\frac{\tilde{\rho}^2 + u^5 + kh(u)^2(ku^2 - 4\tilde{\rho}\alpha + 4k\alpha^2h(u)^2)}{1+f(u)h'(u)^2}}. \tag{22}$$

Equation (21) can be solved numerically. Since we have a family of solutions parametrized by the momentum k, we can look for the one which minimizes the free energy density $\mathcal{F}$, given by

$$\mathcal{F}(\rho) = \mu\rho + \int du \mathcal{L}_E, \tag{23}$$

where $\mathcal{L}_E$ is the DBI Lagrangian plus the Chern-Simons term. Note that the free energy $\mathcal{F}$ is a function of ρ, and not μ. We have identified the momentum with the lowest value of the free energy, and the expectation value of the current operator $\langle\tilde{J}\rangle$ dual to $h(u)$ can be read off from the asymptotic behavior of the normalizable solutions.

So far, we have focused on the dynamics on the D8 brane world volume. The analysis on the $\overline{\text{D8}}$-branes is identical except that the Chern-Simons coupling has the opposite sign due to the *CPT* invariance. There are separate gauge fields A_L and A_R on the D8 and $\overline{\text{D8}}$-branes, respectively, and they cause the instability above the critical charge density. The baryon vector current is dual to $(A_L + A_R)$ and the axial-vector current is dual to $(A_L - A_R)$. The baryon charge density turns on the same amount of chemical potentials for both A_L and A_R. Above the critical baryon density, the instability will take place on both branes, and both vector and axial baryon currents are generated on the boundary. In fact, directions of the momenta on the D8 and $\overline{\text{D8}}$-branes can be different, and the currents J_L and J_R dual to A_L and A_R can carry momenta in different directions.

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