Free Energy Cascade in Gyrokinetic Turbulence

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In gyrokinetic theory, the quadratic nonlinearity is known to play an important role in the dynamics by redistributing (in a conservative fashion) the free energy between the various active scales. In the present study, the free energy transfer is analyzed for the case of ion temperature gradient driven turbulence. It is shown that it shares many properties with the energy transfer in fluid turbulence. In particular, one finds a (strongly) local, forward (from large to small scales) cascade of free energy in the plane perpendicular to the background magnetic field. These findings shed light on some fundamental properties of plasma turbulence, and encourage the development of large-eddy-simulation techniques for gyrokinetics.

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Fully developed turbulence is fundamentally linked to a conservative transfer of (free) energy in wave number space from drive to dissipation scales [1]. While the respective cascade dynamics for simple fluids (described by the Navier-Stokes equation) has been the subject of countless studies and is fairly well understood, the situation is quite different for turbulent plasmas, both at large scales (compared to the particles' gyroradii)—described by magnetohydrodynamics (MHD)—and, in particular, at small scales—described by gyrokinetic theory [2]. The latter case, in which one deals with a gyrocenter distribution function in five-dimensional phase space, shall be the focus of the present work.

In three-dimensional Navier-Stokes turbulence, the kinetic energy is conserved by the advective nonlinearity. It is usually assumed to be injected into the system at the largest scales through mechanical forcing, and to be dissipated at the smallest scales by viscous effects. The role of the nonlinearity is then to transfer the kinetic energy from the large scales to the small ones in what is usually referred to as a cascade process. In the gyrokinetic formalism, on the other hand, the *free energy* acts as the quadratic conserved quantity (see, e.g., Ref. [3] and various references therein). It is usually injected into the system at large scales via the background density and temperature gradients, and expected to be dissipated at small (space and/or velocity space) scales. It is anticipated that one role of the nonlinear term in gyrokinetic turbulence is to transfer the free energy from the largest perpendicular scales to the smallest ones [4–6], but a definitive investigation of the free energy transfer dynamics in a self-driven, three-dimensional system (which is the standard case for magnetically confined plasmas) is still lacking and shall be provided for the first time in the present Letter.

Our study is based on numerical solutions of the nonlinear gyrokinetic equations obtained by means of the GENE code [7–9]. Although GENE is able to treat general toroidal geometry as well as magnetic field fluctuations and collisions, these features shall not be used here. Instead, we will focus on the reduced problem of electrostatic fluctuations, and a large aspect-ratio, circular cross-section model equilibrium. For this simplified case, the respective (appropriately normalized) equations read (for details, see Ref. [9]):

$$\frac{\partial f_j}{\partial t} + \left[\omega_{n_j} + \left(v_\parallel^2 + \mu B_0 - \frac{3}{2}\right)\omega_{T_j}\right] \times F_{0j}\frac{\partial \bar{\phi}_1}{\partial y} + \frac{T_{0j}(2v_\parallel^2 + \mu B_0)}{q_j B_0} \times \left(\mathcal{K}_x \frac{\partial h_j}{\partial x} + \mathcal{K}_y \frac{\partial h_j}{\partial y}\right) + [\bar{\phi}_1, h_j]_{xy} + \frac{v_{Tj}}{2}[v_\parallel^2 + \mu B_0, h_j]_{z v_\parallel} = 0. \quad (1)$$

Here, the total distribution function F_j of species j is split into a Maxwellian part $F_{0j} = \pi^{-3/2} e^{-(v_\parallel^2 + \mu B_0)}$ and a perturbed part f_j, and the nonadiabatic part of f_j is given by $h_j = f_j + (q_j \bar{\phi}_1 / T_{0j}) F_{0j}$ where $\bar{\phi}_1$ is the gyro-averaged electrostatic potential. h_j and f_j depend on the gyrocenter position $\mathbf{r} = (x, y, z)$, the parallel velocity $v_\parallel$, the magnetic moment μ, and the time t. As indicated already above, all simulations in this paper are performed in $\hat{s} - \alpha$ geometry with $\alpha = 0$, for which the curvature terms are given by $\mathcal{K}_x = -2\sin z$ and $\mathcal{K}_y = -2(\cos z + \hat{s} z \sin z)$. Furthermore, $v_{Tj} = (2T_{j0}/m_j)^{1/2}$ is the thermal velocity, $\omega_{n_j} = -R\partial \log n_{0j}/\partial x$ and $\omega_{T_j} = -R\partial \log T_{0j}/\partial x$ are the normalized background density and temperature gradients, R is the major radius, m_j and q_j are the mass and charge of species j. The equilibrium magnetic field is taken to be $B = B_0 B_{\mathrm{ref}}$ where B_{ref} is the reference magnetic field on the magnetic axis. Finally, the Poisson brackets are defined by $[f, g]_{ab} = \partial_a f \partial_b g - \partial_b f \partial_a g$. Note that in Eq. (1), the second term is responsible for the injection of free energy

into the system. The third through fifth terms are, respectively, the curvature, nonlinear, and parallel terms, none of which acts as a source or sink of free energy. Since the simulations presented below are done without collisions, the numerical scheme used in GENE is not dissipative, and a statistical steady state cannot be reached without some form of dissipation [10], hyperdiffusion terms $\mathcal{D}_z f_j$ and $\mathcal{D}_{v_\parallel} f_j$ are added to remove fine-scale fluctuations in z and $v_\parallel$ (for details, see Ref. [11]).

Equation (1) is complemented by the gyrokinetic Poisson equation which is used to determine the self-consistent electrostatic potential:

$$\sum_j \frac{q_j^2 n_{0j}}{T_{0j}}[1-\Gamma_0(b_j)]\phi_1 = \sum_j n_{0j}\pi q_j B_0 \int J_0(\lambda_j) f_j dv_\parallel d\mu. \quad (2)$$

Here, J_0 is the Bessel function and $\Gamma_0(b_j) = e^{-b_j} I_0(b_j)$ with the modified Bessel function I_0. The (dimensionless) arguments b_j and λ_j are defined, respectively, as

$$b_j = \frac{v_{T_j}^2}{2\Omega_j^2} k_\perp^2, \qquad \lambda_j = \frac{v_{Tj}}{\Omega_j}(\mu B_0)^{1/2} k_\perp, \quad (3)$$

where $\Omega_j = (q_j B_0)/(m_j c)$ and $k_\perp$ is the perpendicular wave number.

In the absence of drive and dissipation, the gyrokinetic equations, Eqs. (1) and (2), are known to conserve the free energy $\mathcal{E}$ (see, e.g., Refs. [3,6]) which is usually split into two quadratic parts according to $\mathcal{E} = \mathcal{E}_f + \mathcal{E}_\phi$ with

$$\mathcal{E}_f = \sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} \frac{f_j^2}{2}, \qquad \mathcal{E}_\phi = \sum_j \int d\Lambda q_j \frac{\bar{\phi}_1 f_j}{2}. \quad (4)$$

Here, $\int d\Lambda = \int d^3x \int \pi B_0 n_{0j} dv_\parallel d\mu$ denotes phase-space integration. The evolution equation for the free energy is given by

$$\frac{\partial \mathcal{E}}{\partial t} = \sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} h_j \frac{\partial f_j}{\partial t} = \mathcal{G} - \mathcal{D} \quad (5)$$

in terms of the source term

$$\mathcal{G} = -\sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} h_j \cdot \left[\omega_n + \left(v_\parallel^2 + \mu B_0 - \frac{3}{2}\right)\omega_{Tj}\right] \times F_{0j} \frac{\partial \bar{\phi}_1}{\partial y} \quad (6)$$

and the dissipative term

$$\mathcal{D} = -\sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} h_j (\mathcal{D}_z f_j + \mathcal{D}_{v_\parallel} f_j). \quad (7)$$

The quantity $\mathcal{E}$ plays the same role in gyrokinetic turbulence as the kinetic energy in fluid turbulence [3].

The transfer of free energy between different modes in the saturated turbulent state is induced by the nonlinear term. Although it does not affect the global value of the free energy (numerically, this is satisfied in GENE up to machine precision), it can change, e.g., the value of this quantity associated with particular perpendicular wave numbers. Following the procedure used for studying energy transfer in Navier-Stokes and in MHD turbulence [12–14], we decompose the perpendicular wave vector plane into domains and measure the free energy transfer between these domains. The set of domains $\{d_\ell\}$ is assumed to be a partition (no intersection between the domains and all domains together cover the entire plane). The distribution function and electrostatic potential can then be written as a sum over all contributions for which the perpendicular wave vectors lie in the domain d_ℓ. As a consequence of the Parseval theorem, the free energy can also be split into parts which are associated to the domains d_ℓ: $\mathcal{E} = \sum_\ell \mathcal{E}^\ell = \sum_\ell \mathcal{E}_f^\ell + \sum_\ell \mathcal{E}_\phi^\ell$. In the problem considered hereafter, both the entropy and electrostatic contributions to the free energy are conserved separately by the nonlinearity $\mathcal{N}$. It is thus legitimate to consider the entropy conservation independently from the conservation of the electrostatic energy. The evolution of $\mathcal{E}_f^\ell$ due to the nonlinear term can be expressed as

$$\left.\frac{\partial \mathcal{E}_f^\ell}{\partial t}\right|_{\mathcal{N}} = \sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} f_j^\ell \left.\frac{\partial f_j}{\partial t}\right|_{\mathcal{N}}, \quad (8)$$

where we have used the property $\int d\Lambda f_j^\ell f_j^\ell = \int d\Lambda f_j^\ell f_j$ which is easily proven and expresses the fact that the contributions f_j^ℓ are orthogonal "vectors" if their scalar product is defined as the integration over Λ. Introducing the explicit form of the nonlinearity, one obtains

$$\left.\frac{\partial \mathcal{E}_f^\ell}{\partial t}\right|_{\mathcal{N}} = \sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} f_j^\ell [\bar{\phi}_1, f_j]_{xy} = \sum_{\ell_1,\ell_2} T^{\ell;\ell_1,\ell_2}, \quad (9)$$

where the three-domain interaction terms are defined as

$$T_f^{\ell;\ell_1,\ell_2} = \sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} f_j^\ell [\bar{\phi}_1^{\ell_1}, f_j^{\ell_2}]_{xy}. \quad (10)$$

Equation (9) shows that the evolution of the entropy associated to the domain d_ℓ is the sum of triple interactions between wave vectors associated with the domains d_ℓ, d_{ℓ_1}, and d_{ℓ_2}. This is not a surprise since, like in the Navier-Stokes equation, the quadratic nonlinearity in the gyrokinetic equation is responsible for triadic interactions between the Fourier modes. Proposing a clean definition of the energy transfer between two domains might thus be problematic in such a picture. However, considering the structure of these three-domain interaction terms, the following two-domain interaction terms are natural quantities to investigate:

$$T_f^{\ell,\ell'} = \sum_{\ell_1} T_f^{\ell;\ell_1,\ell'} = \sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} f_j^{\ell}[\bar{\phi}_1, f_j^{\ell'}]_{xy}. \quad (11)$$

These two-domain interaction terms will be interpreted as the energy transfers between the domains d_ℓ and $d_{\ell'}$, even if the redistribution of the free energy between the different domains by the nonlinear term cannot be fully understood without considering triadic interactions. As a consequence of the Poisson bracket structure, it is easy to show that $T_f^{\ell,\ell'} = -T_f^{\ell',\ell}$, which reinforces the interpretation in terms of free energy exchange. Indeed, if the domain d_ℓ is considered to receive a certain amount of free energy per unit of time $T_f^{\ell,\ell'}$ from the domain $d_{\ell'}$, then the domain $d_{\ell'}$ is seen as losing exactly the same amount of free energy per unit of time in profit of the domain d_ℓ. The same approach can be used to define three-domain and two-domain interaction terms for the electrostatic part of the free energy with the following definitions:

$$T_\phi^{\ell;\ell_1,\ell_2} = \sum_j \int d\Lambda \bar{\phi}_1^{\ell}[\bar{\phi}_1^{\ell_2}, f_j^{\ell_1}]_{xy}, \quad (12)$$

$$T_\phi^{\ell,\ell'} = \sum_{\ell_1} T_\phi^{\ell;\ell_1,\ell'} = \sum_j \int d\Lambda \bar{\phi}_1^{\ell}[\bar{\phi}_1^{\ell'}, f_j]_{xy}. \quad (13)$$

The complete dynamical equation for $\mathcal{E}^\ell$ then reads

$$\frac{\partial \mathcal{E}^\ell}{\partial t} = \sum_{\ell'} T_f^{\ell,\ell'} + \sum_{\ell'} T_\phi^{\ell,\ell'} + \mathcal{G}^\ell - \mathcal{D}^\ell, \quad (14)$$

where the source and dissipation terms, $\mathcal{G}^\ell$ and $\mathcal{D}^\ell$, are given by Eqs. (6) and (7), using h_j^ℓ, f_j^ℓ, and $\bar{\phi}_1^\ell$.

The free energy transfer terms defined above are now evaluated from a numerical simulation using GENE. The physical parameters employed in this context correspond to a widely used case of collisionless ion temperature gradient (ITG) turbulence known as the Cyclone Base Case [15] where adiabatic electrons and one single ion species are used. (Repeating the analysis shown below for a reduced normalized temperature gradient of 6.3 instead of the nominal 6.9, or with collisions added, no major changes were observed, indicating the robustness of our findings.) The simulation domain is about 125 ion gyroradii wide in the perpendicular directions, and $256 \times 128 \times 16 \times 48 \times 16$ grid points are used in $(x, y, z, v_\parallel, \mu)$ space. For further analysis, the perpendicular wave vector plane is divided into shells $d_\ell = \{\mathbf{k}_\perp \text{such as} K_\ell < |\mathbf{k}_\perp| \leq K_{\ell+1}\}$ where the shell boundaries K_ℓ are chosen to grow algebraically $K_{\ell+1} = \lambda K_\ell$, with $\lambda = 2^{1/5}$ between shell $\ell = 3$ and $\ell = 24$. The first shell boundaries have been chosen differently ($K_1 = 0$, $K_2 = 0.2$, $K_3 = 0.3$) in order to ensure that enough modes belong to those shells. Moreover, in order to limit the number of shells, the last shell ($\ell = 25$) is wider and limited by $K_{25} = 6.3$ and $K_{26} = |\mathbf{k}_\perp|_{\max} = 14.6$.

Figure 1 shows the numerical results for the source and dissipation terms (averaged over time during the saturated phase of the simulation). As expected, the injection of free energy is well localized at low $k_\perp$. However, as it turns out, the dissipative terms are not just active in the high $k_\perp$ range, but throughout the entire $k_\perp$ spectrum, including the drive range. An explanation of this phenomenon may be provided in terms of the nonlinear coupling to damped eigenmodes, as is discussed in Ref. [16]. There is a net source of free energy up to shell $\ell = 8$ and a net dissipation beyond that.

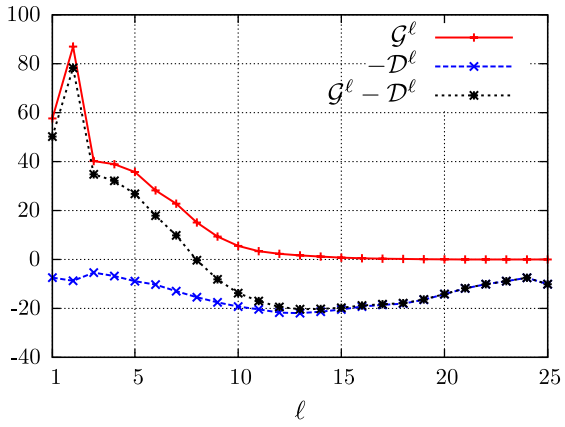

FIG. 1 (color online). Shell decompositions in perpendicular wave number space of the drive ($\mathcal{G}^\ell$) and dissipation ($-\mathcal{D}^\ell$) terms (as well as their sum) from a GENE simulation of ITG turbulence.

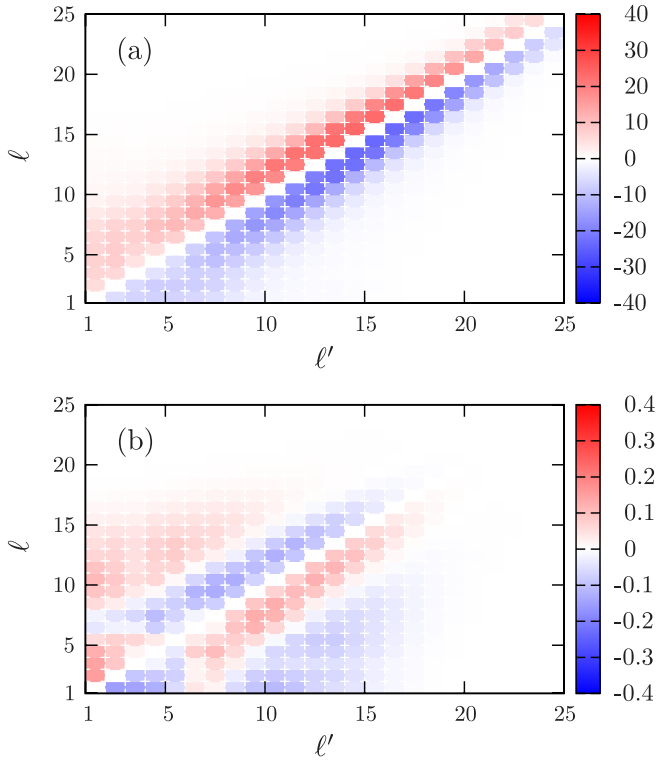

FIG. 2 (color online). Shell-to-shell transfer in perpendicular wave number space of entropy (a) and electrostatic energy (b) from a GENE simulation of ITG turbulence.

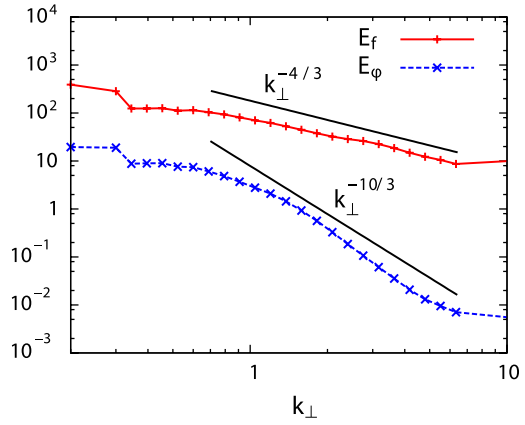

FIG. 3 (color online). Wavenumber spectra of $\mathcal{E}_f$ and $\mathcal{E}_\phi$ from a GENE simulation of ITG turbulence. Predictions from a two-dimensional theory are shown for comparison.

The corresponding shell-to-shell free energy transfer terms are shown in Fig. 2, and various interesting features can be observed there: (i) The entropy transfer is larger than the electrostatic energy transfer by 2 orders of magnitude. This is in line with the fact that only $\mathcal{E}_f$ is driven directly, while $\mathcal{E}_\phi$ is fed via linear transfer terms. Just a small fraction of the free energy is passed on to $\mathcal{E}_\phi$. (ii) While the electrostatic energy exhibits an inverse and less local cascade behavior, this property hardly affects the overall free energy dynamics, given the dominance of $\mathcal{E}_f$ over $\mathcal{E}_\phi$ and of T_f over T_ϕ. (iii) The entropy transfer (and therefore also the free energy transfer) is from large scales to small ones; it is negative for $\ell' > \ell$ and, due to the antisymmetry property, positive otherwise. (iv) The free energy transfer is very local in wave number space. Indeed, only values of $T_{\mathrm{tot}}^{\ell,\ell'}$ with ℓ close to ℓ' are significantly different from zero. In practice, for $|\ell - \ell'| > 5$ the free energy transfers almost vanish. This corresponds to a ratio of wave numbers between the two shells of the order of 2. (v) For $\ell > 15$, the total transfers are found to depend mainly on $\ell - \ell'$, not on the two indices separately. This suggests the existence of an asymptotic self-similarity range, despite finite dissipation (see Fig. 1). Given that, in contrast to the damping rates, the nonlinear frequencies characterizing the free energy transfer increase with ℓ (see also Ref. [16]), cascade dynamics is allowed to develop.

Interestingly, as is shown in Fig. 3, the wave number spectra of $\mathcal{E}_f$ and $\mathcal{E}_\phi$ exhibit power laws at $k_\perp > 1$, indicative of self-similarity. Dimensional analysis based on two-dimensional gyrokinetics lead Schekochihin and coworkers [3] to the predictions $\mathcal{E}_f(k_\perp) \propto k_\perp^{-4/3}$ and $\mathcal{E}_\phi(k_\perp) \propto k_\perp^{-10/3}$ which are displayed for comparison. One finds that the measured spectra are relatively close to these expectations, regardless of the fact that terms related to parallel free streaming, magnetic curvature, and inherent drive and damping are all neglected in this theory, and that $\mathcal{E}_f$ and $\mathcal{E}_\phi$ are conserved independently of each other in two dimensions. Clearly, future work will have to further unravel the underlying physics.

In summary, the spectral transfer of free energy in gyrokinetic turbulence displays various similarities with the kinetic energy transfer in fully developed Navier-Stokes turbulence, although this is not at all obvious *a priori*. In particular, being dominated by the entropy contribution, the free energy is subject to a (strongly) local, forward cascade—despite the absence of a strict inertial range. Moreover, the wave number spectra of the entropic and electrostatic parts of the free energy exhibit power laws with exponents which are close to the predictions from a simplified two-dimensional analysis.

Insights like these may be expected to guide the application of large-eddy-simulation techniques [17] to gyrokinetics. Here, the idea is to only retain the dynamics of the largest scales while the smallest ones are modeled. Indeed, if the smallest scales are proven to act systematically as a sink of free energy like it was the case here, it is reasonable to propose a dissipative model for these small scales and consequently reduce as much as possible the numerical resolution. On such a basis, it may well become possible to reduce the computational effort for gyrokinetic turbulence simulations by a significant amount. The present work represents a relevant step in that direction.

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[1] G. Falkovich and K. R. Sreenivasan, Phys. Today **59**, No. 4, 43 (2006).
[2] A. J. Brizard and T. S. Hahm, Rev. Mod. Phys. **79**, 421 (2007)
[3] A. A. Schekochihin *et al.*, Astrophys. J. **182**, 310 (2009).
[4] G. G. Howes *et al.*, Phys. Rev. Lett. **100**, 065004 (2008).
[5] T. Tatsuno *et al.*, Phys. Rev. Lett. **103**, 015003 (2009).
[6] T. Tatsuno *et al.*, J. Plasma Fusion Res. SERIES **9**, 509 (2010).
[7] F. Jenko *et al.*, Phys. Plasmas **7**, 1904 (2000).
[8] T. Dannert and F. Jenko, Phys. Plasmas **12**, 072309 (2005).
[9] F. Merz, Ph.D. thesis, Universität Münster, 2009.
[10] J. A. Krommes and G. Hu, Phys. Plasmas **1**, 3211 (1994).
[11] M. J. Pueschel *et al.*, Comput. Phys. Commun. **181**, 1428 (2010).
[12] O. Debliquy *et al.*, Phys. Plasmas **12**, 042309 (2005).
[13] A. Alexakis, P. D. Mininni, and A. Pouquet, Phys. Rev. E **72**, 046301 (2005).
[14] D. Carati *et al.*, J. Turbul. **7**, N51 (2006).
[15] A. M. Dimits *et al.*, Phys. Plasmas **7**, 969 (2000).
[16] D. R. Hatch *et al.* (to be published).
[17] O. Agullo *et al.*, Phys. Plasmas **8**, 3502 (2001).