Comment on "Apollonian Networks: Simultaneously Scale-Free, Small World, Euclidean, Space Filling, and with Matching Graphs"

Andrade *et al.* [1] reported that the cumulative degree distribution of an infinite Apollonian network has a power-law tail $k^{1-\gamma}$ with $\gamma = 1 + \ln 3/\ln 2 \equiv \gamma_{\rm PRL}$. In a recent erratum [2] the authors retracted this result and asserted that $\gamma = \ln 3/\ln 2 \equiv \gamma_{\rm ERR}$. Since for scale-free networks the average degree diverges when $1 < \gamma_{\rm ERR} < 2$, while for $2 < \gamma_{\rm PRL} < 3$ it remains finite, this effects drastically the nature of phase transitions in spin models on such networks [1,3,4]. For the Apollonian network the values of k for which the degree distribution is nonzero are highly sparse for large k. By properly taking sparsity into account, an equivalent distribution for all k yields a power-law tail $p(k) \sim k^{-\gamma}$ with $\gamma = \gamma_{\rm PRL}$. Therefore the result in [1] is actually correct and the Erratum is wrong.

Denote by $N_n^{(j)}$ and $k_n^{(j)}$, the number and degree of vertices at generation n, given they were created at generation j ($j \leq n$),

$$(N_n^{(j)}, k_n^{(j)}) = \begin{cases} (3, 2^n + 1) & j = 0, \\ (3^{j-1}, 3 \times 2^{n-j}) & 1 \leq j \leq n. \end{cases} \quad (1)$$

Note that the expression for $k_n^{(0)}$ in the erratum is still wrong. Using Eq. (1), the cumulative degree distribution $P(k_n^{(j)}) = (\sum_{i=0}^{j} N_n^{(i)})/N_n$ scales as

$$P(k_n^{(j)}) \sim 3^{j-n} \sim (k_n^{(j)})^{-\ln 3/\ln 2} \equiv (k_n^{(j)})^{1-\gamma}, \quad (2)$$

where the last equality is the author's definition of γ [1]. Thus $\gamma = \gamma_{\rm PRL}$ and the Erratum [2] is clearly wrong. Likewise, from Eq. (1), the first moment of the degree distribution for large n approaches $\langle k \rangle = 6$, while the second moment diverges as $\langle (k_n)^2 \rangle \sim (4/3)^n \sim k_{\rm max}^{3-\gamma}$, where $k_{\rm max} = k_n^{(1)}$ is the cutoff and again $\gamma = \gamma_{\rm PRL}$. Both results are compatible with a power-law distribution $2 < \gamma_{\rm PRL} < 3$.

On the other hand from Eq. (1) when n is large, $p_{k_n^{(j)}} \equiv N_n^{(j)}/N_n \rightarrow 2 \times 3^{(j-n)}/3$ so that

$$p_{k_n^{(j)}} \sim (k_n^{(j)})^{-\gamma_{\rm ERR}}, \quad (3)$$

implying that the cumulative and marginal distributions, Eqs. (2) and (3), scale with degree in the same way and with exponent $\gamma = \gamma_{\rm ERR}$. This seems inconsistent, since the average degree is finite while its variance diverges.

The problem is that writing the distribution as in Eq. (3) is misleading, since the set of permissible k values, Eq. (1), is sparse and nonuniform, as also noted before [5]. This is already implicit in the calculations of the first and second moments which are sums over j and thus restrict the values of k used. As this example clearly shows, an exponent for a power-law distribution is only meaningful if specified along with the domain on which it takes nonzero values. If one wants to extend the degree distribution to the natural domain of all non-negative integers $\mathbb{N}_0$ one has to smoothen the distribution. The probability mass $p_{k_n^{(j)}}$ has then to be distributed in some way, $p(k)$, over a suitably chosen partition of $\mathbb{N}_0$ into intervals $I_{n,j}$ of k values containing $k_n^{(j)}$ so that $\sum_{k \in I_{n,j}} p(k) = p_{k_n^{(j)}}$. The widths of these intervals scale as

$$\Delta k_n^{(j)} = (k_n^{(j)} - k_n^{(j-1)}) \sim k_n^{(j)}. \quad (4)$$

Using Eq. (3), $p(k)$ must be of the form

$$p(k) \sim \frac{p_k}{\Delta k} \sim k^{-\gamma_{\rm ERR}-1} \sim k^{-\gamma_{\rm PRL}}, \quad (5)$$

and the scaling behavior of the moments at large k remains the same. By construction, the smoothened degree distribution $p(k)$ when binned with the intervals $I_{n,j}$ will reproduce Eq. (3).

Such smoothened distributions will generally not capture exact values of the moments, but they will correctly describe the scaling behavior of the tails of the distribution, which ultimately controls whether different moments diverge, and if so, how this divergence scales with k cutoff.

The author thanks Nihat Berker for comments and suggestions.

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Received 16 April 2010; published 14 January 2011
DOI: 10.1103/PhysRevLett.106.029802
PACS numbers: 89.75.Hc, 64.60.A−, 89.20.Hh, 89.75.Da

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