Exact Solutions in 3D New Massive Gravity

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We show that the field equations of new massive gravity (NMG) consist of a massive (tensorial) Klein-Gordon-type equation with a curvature-squared source term and a constraint equation. We also show that, for algebraic type D and N spacetimes, the field equations of topologically massive gravity (TMG) can be thought of as the "square root" of the massive Klein-Gordon-type equation. Using this fact, we establish a simple framework for mapping all types D and N solutions of TMG into NMG. Finally, we present new examples of types D and N solutions to NMG.

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The continuing search for a consistent theory of quantum gravity has stimulated many investigators to explore gravity models in three dimensions, where one may hope to have less austere ultraviolet (UV) divergences in perturbation theory. Ordinary general relativity (GR) in three dimensions becomes dynamically trivial as it does not propagate any physical degrees of freedom [1]. Remarkably, this problem can be cured by a particular extension of GR. There exist two popular approaches to such an extension. (i) The Einstein-Hilbert (EH) action is supplemented with a parity-violating gravitational Chern-Simons term. The resulting theory is known as topologically massive gravity [2,3]. (ii) The EH action is extended by adding a particular higher-derivative correction term to it. This gives rise to a novel theory of three-dimensional massive gravity, known as new massive gravity [4].

In contrast to topologically massive gravity (TMG), new massive gravity is a parity-preserving theory. At the linearized level it becomes equivalent to the Fierz-Pauli theory for a free massive graviton in three dimensions, thereby sharing its unitary property [4] (see also [5–7]). It was argued that the theory in its pure quadratic curvature limit is both unitary and power-counting UV finite [6], thereby violating standard paradigm of its "cousins" in four dimensions [8]. Further developments include exact solutions to new massive gravity (NMG). In particular, it was shown that Banados-Teitelboim-Zanelli (BTZ) [9] and warped anti-–de Sitter (AdS_3) black hole solutions of TMG persist in cosmological NMG as well [4,10]. The general AdS_3-wave solution of NMG was found in [11] and Bianchi type IX homogeneous space solutions were studied in [12].

An analysis of NMG in the context of the AdS_3/CFT_2 correspondence reveals the bulk-boundary unitarity conflict: the unitarity in the bulk implies a negative central charge for the boundary conformal field theory, CFT_2 [13–15]. In TMG, with the "right" sign EH term in the action, this conflict is resolved at a "chiral" point, at which the Compton wavelength of the massive graviton becomes equal to the radius of the AdS_3 space [16]. However, for NMG a similar strategy of the chiral point shows that both the energy of massive bulk modes and the central charges of the dual CFT_2 vanish [14,15]. Thus, the theory becomes trivial at this point under the standard Brown-Henneaux boundary conditions [17] (see also [18–21] for some further investigations). In light of all these developments, it is of great importance to undertake an exhaustive study of exact solutions to NMG in hope of finding a stable vacuum for a consistent theory of quantum gravity in three dimensions.

The purpose of this Letter is to give a novel description of NMG in terms of a first-order differential operator (appearing in TMG) acting on the traceless Ricci tensor and to establish a simple framework for mapping all Petrov-Segre types D and N solutions of TMG into NMG. We also present new examples of exact solutions which are only inherent in NMG. These are type D solutions with constant scalar curvature (Bianchi types VI_0 and VII_0 solutions), a type D solution with nonconstant scalar curvature (a new extremal black hole type solution) as well as a type N solution that belongs to a class of Kundt spacetimes.

We begin by recalling the field equations of NMG [4]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0, \tag{1}$$

where $R = g^{\mu\nu} R_{\mu\nu}$ is the three-dimensional Ricci scalar, λ is a cosmological parameter, m is a mass parameter, and $K_{\mu\nu}$ is a symmetric and covariantly conserved tensor given by

$$K_{\mu\nu} = 2\nabla^2 R_{\mu\nu} - \frac{1}{2}(\nabla_\mu \nabla_\nu R + g_{\mu\nu} \nabla^2 R) - 8 R^\alpha_\mu R_{\nu\alpha} + \frac{9}{2} R R_{\mu\nu} + g_{\mu\nu}\left(3 R_{\alpha\beta} R^{\alpha\beta} - \frac{13}{8} R^2\right). \tag{2}$$

Here ∇_μ is the covariant derivative operator with respect to the spacetime metric and $\nabla^2 = \nabla_\mu \nabla^\mu$. The trace of Eq. (1) gives

$$R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2 + m^2 R = 6m^2\lambda. \tag{3}$$

We also recall the field equations of TMG [2,3],

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \tag{4}$$

where Λ is the cosmological constant, μ is a mass parameter, and $C_{\mu\nu}$ is the Cotton tensor,

$$C_{\mu\nu} = \epsilon_{\mu}^{\alpha\beta}\nabla_\alpha\left(R_{\nu\beta} - \frac{1}{4}g_{\nu\beta}R\right), \tag{5}$$

which is a symmetric, traceless and covariantly conserved quantity. Here $\epsilon_{\mu\alpha\beta}$ is the Levi-Civita tensor given by the relation $\epsilon_{\mu\alpha\beta} = \sqrt{-g}\varepsilon_{\mu\alpha\beta}$, $\varepsilon_{012} = 1$.

The trace of Eq. (4) yields

$$R = 6\Lambda. \tag{6}$$

Using the traceless Ricci tensor,

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R, \tag{7}$$

one can pass to the alternative form of the field equations in (4). We have

$$S_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0. \tag{8}$$

We now introduce a first-order differential operator $\not{D}$, whose action on a symmetric tensor $\Phi_{\mu\nu}$ is given by

$$\not{D}\Phi_{\mu\nu} = \frac{1}{2}(\epsilon_{\mu}{}^{\alpha\beta}\nabla_\beta\Phi_{\nu\alpha} + \epsilon_{\nu}{}^{\alpha\beta}\nabla_\beta\Phi_{\mu\alpha}). \tag{9}$$

It is not difficult to show that this expression reduces to

$$\not{D}\Phi_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\nabla_\beta\Phi_{\nu\alpha}, \tag{10}$$

provided that

$$\nabla^\nu\Phi_{\mu\nu} = \nabla_\mu\Phi, \tag{11}$$

where Φ is the trace of the tensor $\Phi_{\mu\nu}$. Choosing this tensor as

$$\Phi_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R, \tag{12}$$

for which condition (11) is fulfilled, and comparing Eqs. (5), (9), and (10) we find that

$$C_{\mu\nu} = -\not{D}\Phi_{\mu\nu} = -DS_{\mu\nu}. \tag{13}$$

Taking this into account in Eq. (8), we obtain

$$\not{D}S_{\mu\nu} = \mu S_{\mu\nu}. \tag{14}$$

It is interesting to note that this equation closely resembles the Dirac equation $\not{D}\Psi_A = \gamma_A^{\mu B}\nabla_\mu\Psi_B = \mu\Psi_A$. Here the Dirac type operator $\not{D}$ acts on the traceless Ricci tensor, $\not{D}S_{\mu\nu} = D_{(\mu\nu)}{}^{(\alpha\beta)\rho}\nabla_\rho S_{\alpha\beta}$. Clearly, the action of $\not{D}$ on Eq. (14) leads to the second-order Klein-Gordon-type equation

$$(\not{D}^2 - \mu^2)S_{\mu\nu} = 0. \tag{15}$$

Next, we rewrite the field equations of NMG given in (1) in terms of the traceless Ricci tensor $S_{\mu\nu}$ and the operator $\not{D}$. Using the fact that the Cotton tensor (5) satisfies relation (11) and taking into account Eqs. (10) and (13), we obtain that

$$\not{D}^2 S_{\mu\nu} = -\epsilon_{\mu}{}^{\alpha\beta}\nabla_\beta C_{\nu\alpha}. \tag{16}$$

It is straightforward to show that this expression, with Eqs. (5) and (12) in mind, can be written in the form

$$\not{D}^2 S_{\mu\nu} = \nabla^2\Phi_{\mu\nu} - \nabla^\rho\nabla_\nu\Phi_{\mu\rho}. \tag{17}$$

This equation can be transformed further by using condition (11) and the standard relation between the Riemann and the Ricci tensors in three dimensions. Using then the resulting expression in Eq. (1), we reduce it into the form of the massive (tensorial) Klein-Gordon-type equation with a curvature-squared source term. Thus, we obtain

$$(\not{D}^2 - m^2)S_{\mu\nu} = T_{\mu\nu}, \tag{18}$$

where the traceless source term is given by

$$T_{\mu\nu} = S_{\mu\rho}S_\nu^\rho - \frac{R}{12}S_{\mu\nu} - \frac{1}{3}g_{\mu\nu}S_{\alpha\beta}S^{\alpha\beta}. \tag{19}$$

Meanwhile, the trace equation in (3) takes the form

$$S_{\mu\nu}S^{\mu\nu} + m^2 R - \frac{R^2}{24} = 6m^2\lambda. \tag{20}$$

We note that in this description, the field equations NMG reduce to two independent equations (18) and (20), where the latter equation can be thought of as a constraint. In the canonical description, Eq. (3) is a consequence of (1). It is easy to see that with the relation

$$T_{\mu\nu} = \kappa S_{\mu\nu}, \tag{21}$$

where κ is a function of the scalar curvature, which is fulfilled for algebraic types D and N spacetimes, Eq. (18) takes the form

$$(\not{D}^2 - \mu^2)S_{\mu\nu} = 0. \tag{22}$$

Here

$$\mu^2 = m^2 + \kappa. \tag{23}$$

Comparing this equation with that in (15) we see that, for type N spacetimes and type D spacetimes with constant scalar curvature, they become equivalent to each other. That is, in the case under consideration, the field equations of TMG in (14) can be thought of as the square root of those of NMG given in (22). This fact enables us to map all types D and N solutions of TMG into NMG.

We first focus on algebraic type D solutions. It turns out that for every algebraic type D solution of TMG there exist two inequivalent type D solutions to NMG, provided that the solution parameters are related by

$$\mu^2 = \frac{9m^2}{7}\left(2 \pm \frac{\sqrt{5 + 7\lambda/m^2}}{\sqrt{3}}\right), \tag{24}$$

$$\Lambda = -\frac{2m^2}{21}\left(13 \pm \frac{10\sqrt{5 + 7\lambda/m^2}}{\sqrt{3}}\right). \tag{25}$$

The proof of this statement is straightforward. We recall that type D spacetimes in three dimensions are split into types D_t and D_s, depending on whether the one-dimensional eigenspace of $S^\mu{}_\nu$ is timelike or spacelike (see, for instance, [22]). For type D_t spacetimes the canonical form of the traceless Ricci tensor $S_{\mu\nu}$ is given by

$$S_{\mu\nu} = p(g_{\mu\nu} + 3t_\mu t_\nu), \tag{26}$$

where p is a scalar function and t_μ is a timelike vector normalized as $t_\mu t^\mu = -1$. In works [22,23], it was shown that when p is constant and t_μ is a Killing vector, obeying the equation

$$\nabla_\nu t_\mu = \frac{\mu}{3}\epsilon_{\mu\nu\sigma}t^\sigma, \tag{27}$$

the field equations of TMG are solved, fixing the value of p. With this in mind, substituting (26) in (19) and (21) we find that

$$\kappa = -p - \frac{R}{12}. \tag{28}$$

For any vector with $\nabla_\mu k^\mu = 0$, we have

$$\nabla^\mu\nabla_\nu k_\mu = R_\nu{}^\sigma k_\sigma = -\left(2p - \frac{R}{3}\right)k_\nu, \tag{29}$$

where in the last step we have used Eqs. (7) and (26). From Eqs. (27) and (29) it follows that

$$6p = \frac{2}{3}\mu^2 + R. \tag{30}$$

Combining now this equation with those in (23) and (28), and taking into account Eq. (6), we obtain

$$m^2 - \frac{10}{9}\mu^2 = \frac{3}{2}\Lambda. \tag{31}$$

Meanwhile, substitution of (26) into Eq. (20), with Eqs. (6) and (30) in mind, yields

$$\frac{1}{6}\left(\frac{2}{3}\mu^2 + 6\Lambda\right)^2 + 6m^2\Lambda - \frac{3}{2}\Lambda^2 = 6m^2\lambda. \tag{32}$$

Solving algebraic equations (31) and (32), we arrive at the relations given in (24) and (25). A similar analysis shows that these relations remain unchanged for type D_s spacetimes.

We now show that every algebraic type N spacetime of TMG provides two inequivalent type N solutions to NMG and the parameters of the solutions are adjusted according to the relations

$$\mu^2 = \mp m^2\sqrt{1 - \lambda/m^2}, \tag{33}$$

$$\Lambda = 2m^2(1 \pm \sqrt{1 - \lambda/m^2}). \tag{34}$$

For spacetimes of type N, the canonical form of the traceless Ricci tensor is given by

$$S_{\mu\nu} = l_\mu l_\nu, \tag{35}$$

where l_μ is a null vector [24]. Substitution of this tensor into Eqs. (20) and (21) yields

$$\Lambda(4m^2 - \Lambda) = 4m^2\lambda \tag{36}$$

and $\kappa = -\Lambda/2$, respectively. Hence, Eq. (23) takes the form

$$m^2 - \mu^2 = \frac{1}{2}\Lambda, \tag{37}$$

Again, solving algebraic Eqs. (36) and (37) we obtain the relations given in (33) and (34).

The novel description of NMG also turns out to be a very powerful tool for finding new exact types D and N solutions to NMG which do not have their counterparts in TMG. A crucial step on this route amounts to the following: In TMG, as seen from Eq. (14), the action of the operator $\not{D}$ on the traceless Ricci tensor $S_{\mu\nu}$ results in the same type geometry. Meanwhile, this is not the case in general, where the same operation leads to "intermediate" geometries as well. However, the secondary action of $\not{D}$ restores the original types D and N geometries in NMG. Furthermore, it turns out that $p = -R/3$ for type D solutions with constant scalar curvature. Altogether, these facts enable us to find all type D and N solutions of NMG. Below, we present some intriguing and simple examples of such solutions.

We begin with the type D_t solution with constant scalar curvature which is given by

$$ds^2 = -dt^2 + e^{2\sqrt{2/5}mt}dx^2 + e^{-2\sqrt{2/5}mt}dy^2 \tag{38}$$

and $\lambda = m^2/5$. This solution admits a three-parameter group of motions and the associated Killing vectors form the Lie algebra

$$[\xi_1, \xi_2] = 0, \qquad [\xi_1, \xi_3] = \sqrt{2/5}m\xi_1, \qquad [\xi_2, \xi_3] = -\sqrt{2/5}m\xi_2. \tag{39}$$

Thus, the solution is of a homogeneous anisotropic spacetime of Bianchi type VI_0, or with $E(1,1)$ symmetry.

There also exists the type D_s constant curvature solution given by

$$ds^2 = \cos(2\sqrt{2/5}mx)(-dt^2 + dy^2) + dx^2 + 2\sin(2\sqrt{2/5}mx)dtdy \quad (40)$$

and $\lambda = m^2/5$. This is a homogeneous anisotropic spacetime of Bianchi type VII_0, or with $E(2)$ symmetry, which can be obtained from (38) by an appropriate analytical continuation as well.

It turns out that for $\mu^2 = 0$ in (23) (and only for this case [25]) NMG gravity also admits a class of type D solutions with nonconstant scalar curvature. For these solutions, as seen from Eq. (20), $\lambda = m^2$. The most simple example (type D_s) with one Killing vector is given by

$$ds^2 = -k^2(r - r_0)^2dt^2 + \frac{dr^2}{k^2(r - r_0)^2} + [f(\phi) + r - r_0]^2 d\phi^2, \quad (41)$$

where $f(\phi)$ is an arbitrary function and $k^2 = -2m^2$ ($m^2 < 0$). This is an extremal black hole type solution with the horizon located at $r = r_0$ and the surface gravity is zero. In the asymptotic region $r \to \infty$, it becomes the metric of AdS_3 spacetime, whereas in the near-horizon region $r \to r_0$, we have the metric of $\mathrm{AdS}_2 \times S^1$. For $f(\phi) = \mathrm{const}$ this solution describes an extremal black hole which was earlier found in [13]. Finally, we present a new remarkably simple type N solution given by

$$ds^2 = d\rho^2 + 2\cosh^2(\nu\rho)dudv + \cosh(\nu\rho) \times [\cosh(\mu\rho)f(u) - v^2\nu^2\cosh(\nu\rho)]du^2, \quad (42)$$

and $\lambda = -\nu^2(1 + \nu^2/4m^2)$. Here $f(u)$ is an arbitrary function and $\nu = \sqrt{-\Lambda}$. This metric does not admit the null Killing vector and belongs to a class of Kundt spacetimes. However, when $\Lambda \to 0$, the Killing isometry appears and ∂_v becomes a covariantly constant null Killing vector. The resulting metric represents pp waves being the limiting case of AdS pp waves found in [11].

In summary, the results presented in this Letter are of interest for several reasons: First of all, we have given a novel description of NMG in terms of a first-order differential operator, appearing in TMG and resembling a Dirac type operator, acting on the traceless Ricci tensor. Such a description has a striking consequence, greatly simplifying the search for all types D and N solutions to NMG. This is a very large class of solutions and their exhaustive exploration is given in [25,26]. Here we have established a simple framework, involving in essence an algebraic procedure, for mapping all types D and N exact solutions of TMG into NMG. We have also presented some intriguing and the most simple examples of types D and N new exact solutions to NMG, which do not have their counterparts in TMG.

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