Perfect Quantum Routing in Regular Spin Networks

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Motivated by the need for quantum computers to communicate between multiple, well separated qubits, we introduce the task of quantum routing for distributing quantum states, and generating entanglement, between these sites. We describe regular families of coupled quantum networks which perfectly route qubits between arbitrary pairs of nodes with a high transmission rate. The ability to route multiple states simultaneously and the regularity of the networks vastly improve the utility of this scheme in comparison to the task of state transfer, leading us to propose an implementation in optical lattices.

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Introduction.—The task of quantum state transfer was introduced in the context of quantum computation as a protocol to simplify interactions between distant qubits in an architecture that has locality restrictions, as in solid state systems. This study was initiated by Bose, who analyzed a uniformly coupled quantum chain and evaluated its efficacy for transferring an unknown quantum state from one end to the other [1]. A plethora of protocols have since been introduced to achieve transfer perfectly [2–5] or with arbitrary accuracy [6]. Even without experimental implementation, these studies have proven extremely useful as a constructive tool [7,8].

Various factors have severely reduced the feasibility of existing state transfer protocols. All known perfect transfer protocols that do not directly couple every qubit in the network [9] (and hence do not have trivial transfer distance) transfer an input quantum state from a given site onto a specific, corresponding output site, and can never significantly enhance the connectivity of a scaling computational system. Furthermore, the solutions are either highly nonlocal (hypercubes [10–12] and integral circulant graphs [13]), or local but irregular. These can be circumvented by introducing some control on the input and output spins [14–16]. However, some level of control was tacitly assumed in *all* scenarios, in that quantum states must be introduced and removed from the input and output spins of the network, which typically must be implemented quickly in comparison to the Hamiltonian dynamics (although the required speed can be reduced [17,18]). Explicitly acknowledging this additional control produces vastly richer dynamics [19].

In a system which should transfer quantum states between arbitrary input and output ports, we would require control over all possible input and output ports. Hence, in this Letter, we assume that local gates can be applied quickly on these nodes, and use this to design protocols to transfer a quantum state from any node to any other node in a regular network of arbitrary spatial dimension. We refer to this task as perfect routing. In doing so, we overcome the major limitations of previous perfect state transfer schemes and introduce some robustness to manufacturing imperfections. There are also advantages over arbitrarily accurate schemes in terms of scaling properties of the required fields, and the rate of transfer. The same controls can also be used to produce entangled states across the network. In trade for these many advantages, there are still some stringent manufacturing requirements since our construction will make heavy use of symmetry properties which need to be present in the system. Our strategy is to decompose our construction into two steps. The first, which we demonstrate in 1D, is how to take the analysis of a small region of a network, and be able to tile it so the overall Hamiltonian has a direct sum structure, each component of which is the same as the original region you were interested in. Then we show how to analyze these small regions to create a basic routing unit. Finally, we present a possible implementation of this scheme in optical lattices.

One-dimensional prototype.—We start by considering routing in a one-dimensional system of $3N + 1$ qubits, as depicted in Fig. 1(a), in order to illustrate some of the basic ideas of our construction. Consider a Hamiltonian

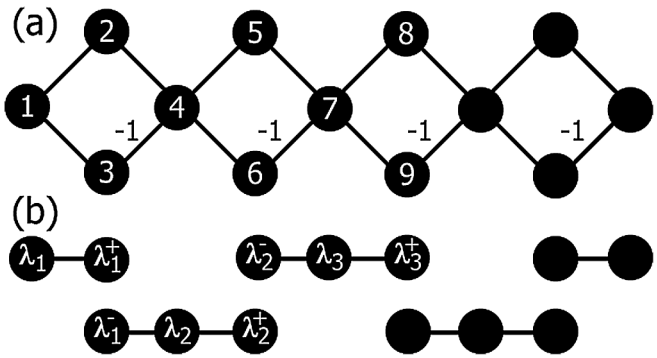

FIG. 1. (a) A quasi-1D routing structure. The circles represent qubits, and the lines indicate an XX coupling between pairs of qubits of strength $+1$, unless -1 is indicated. (b) Under a basis transformation, a simple direct sum structure is apparent. In this case, all coupling strengths are $\sqrt{2}$.

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$$H = \frac{1}{2}\sum_{\{n,m\}\in E} J_{n,m}(X_nX_m + Y_nY_m),$$

where E is the set of edges of the graph depicted in Fig. 1(a), $J_{3k,3k+1} = -1$ for all k and $J_{n,m} = 1$ otherwise ($\hbar = 1$ such that times and energies can be treated as dimensionless). The Hamiltonian is spin preserving,

$$\left[H, \sum_{n=1}^{3N+1} Z_n\right] = 0,$$

so we analyze a protocol where all spins are initialized in the $|0\rangle$ state. Although we have chosen this specific form of Hamiltonian, minor modifications of the present constructions allow the generalization to any local, spin-preserving Hamiltonian such as the anisotropic Heisenberg model or arrays of harmonic oscillators.

The state to be transferred is a superposition of 0 and 1 excitations. Since the 0 excitation subspace is a single state, it is invariant under the Hamiltonian evolution, and we concentrate on the single excitation subspace. The standard, spin, basis is denoted by $|\underline{n}\rangle = |0\rangle^{\otimes n-1}|1\rangle|0\rangle^{\otimes 3N+1-n}$. However, this is not the most natural basis for us to work in. Instead, we define $|\lambda_n\rangle = |\underline{3n+1}\rangle$ and $|\lambda_n^{\pm}\rangle = (|\underline{3n+2}\rangle \pm |\underline{3n+3}\rangle)/\sqrt{2}$, which allows a crucial observation: the action of the Hamiltonian H causes the states $|\lambda_n^{\pm}\rangle$ to interfere destructively on spins $3n+4$ and $3n+1$, respectively. Hence, the Hamiltonian is left with a direct sum structure as depicted in Fig. 1(b) where each subsystem is a uniformly coupled chain of length 2 or 3, and achieves perfect transfer in times $\pi/(2\sqrt{2})$ and $\pi/2$, respectively; i.e., $e^{-iH\pi/2}|\lambda_{n-1}^-\rangle = |\lambda_n^+\rangle$. Now observe that a fast application of the local rotations (but globally applied) $U = \prod_{n=1}^{N} Z_{3n}$ performs the conversion $U|\lambda_n^{\pm}\rangle = |\lambda_n^{\mp}\rangle$, i.e., it transfers the state from one subsystem to the next. Hence, starting from $|\lambda_n^+\rangle$, we apply U every $\pi/2$ and after $|m-n|\pi/2$ we arrive in the state $|\lambda_m^+\rangle$. This achieves the long-range transfer. We now just have to show how to convert from an input state, $|\underline{3n+1}\rangle$ or $|\underline{3n+2}\rangle$, to $|\lambda_n^+\rangle$ in order to be able to use the long-range transfer protocol. This step also has to be inverted at the end, but the periodicity of the dynamics ensure this.

If the input qubit is 1, then $|\lambda_1^+\rangle$ is simply produced by letting the Hamiltonian evolve for time $\pi/(2\sqrt{2})$. This operation is its own inverse. For other starting qubits $3n+1$, we observe that $Z_{3n+2}Z_{3n+3}$ is equivalent to applying a Z rotation to the λ_n^+ end of the 3-qubit effective chain in Fig. 1(b), which can be composed with the natural Hamiltonian evolution to create the necessary transformation. More crucially for the coming generalization, we must show how to start from the vertices of the diamonds, say $|\underline{3n+2}\rangle$, and produce $|\lambda_n^+\rangle$. Provided $n \neq 0, N-1$, we can simply evolve the Hamiltonian for a time $\pi/2$, at which point we have

$$(|\lambda_{n-1}^-\rangle + |\lambda_{n+1}^+\rangle)/\sqrt{2}. \quad (1)$$

Applying phase gates $\sqrt{Z_{3n-1}Z_{3n}}$ and allowing evolution for another time step $\pi/2$ produces $(i|\lambda_n^+\rangle + |\lambda_n^-\rangle)/\sqrt{2}$ and a phase gate $\sqrt{Z_{3n+3}}$ yields $|\lambda_{3n+2}\rangle$.

This proves that we can perfectly propagate an unknown quantum state along the length of the chain between any set of nodes in a time proportional to the distance between the nodes, with the help of local magnetic fields. A bonus is that one of the intermediate states, Eq. (1), is entangled, so we can also produce and distribute Bell pairs. The Hamiltonian and controls are all spin preserving, so without external initialization of the system in some complicated state of many excitations, its effects can be classically simulated (i.e., presumably not universal for quantum computation). Since the subsystems in Fig. 1(b) are independent, we can have multiple excitations in the system simultaneously, provided they are each separated by at least one subsystem. A 1D structure imposes that one state can never move past a second one. To resolve this, we need to move to a two or three dimensional structure, as we will now demonstrate.

Designing systems with a subsystem reduction.—Figure 2 demonstrates the straightforward generalization of the 1D results so that we can construct a basic d-dimensional subsystem building block which we can tile to create the complete network. We simply take the extremal links in the subsystem network, which are coupled with strength J, and replace them with a V structure such that each coupling strength has modulus $J/\sqrt{2}$. These couplings are then patterned so that when the structure is tiled, one of the 4 coupling strengths in each diamond is $-J/\sqrt{2}$. This is depicted for 2D, but works for any dimension d. As before, applying a Z rotation on one of the extremal spins of a block hops an excitation present across the V between $(|01\rangle + |10\rangle)$ and $(|01\rangle - |10\rangle)$, which are effective single excitations on two independent subsystems. Thus, we simply have to demonstrate how to route from any extremal node of the subsystem to any other. Similarly to the 1D case, multiple states can be transferred at once, since all states are kept confined to individual sections; it suffices to keep these sections

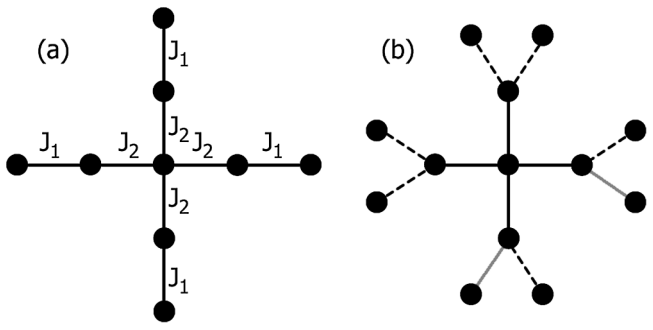

FIG. 2. The basic scheme for a 2D square lattice. Once perfect routing between extremal nodes in (a) has been shown, this can be converted into a repeating unit (b) which can be tiled to give a complete network with a direct sum structure given by (a). Dashed edges denote a scaling factor of coupling strengths by $1/\sqrt{2}$ relative to (a). The gray lines are scaled by $-1/\sqrt{2}$.

separated by a single unit, and then the states never meet and never interfere.

Dynamics within a subsystem.—We will now give a simple construction of a subsystem structure based on the design of perfect state transfer chains. We start from a perfect state transfer chain of $M \geq 5$ qubits (M odd). Such a scheme can be written as

$$H_{\text{chain}} = \sum_{n=1}^{M-1} K_n(|\underline{n}\rangle\langle\underline{n+1}| + |\underline{n+1}\rangle\langle\underline{n}|),$$

and exhibits perfect transfer in time t_0, i.e., $e^{-iH_{\text{chain}}t_0}|\underline{n}\rangle = |\underline{M+1-n}\rangle$. Using the techniques introduced in [20,21], this can be redesigned into a star topology of $2d$ branches, where the central coupling becomes $J_{(M-1)/2}/\sqrt{d}$ (see Fig. 3). Under this transformation, the single excitation states of the chain, $|\underline{n}\rangle$, map to $|W_0^n\rangle$ [for $n \neq (M+1)/2$] where

$$|W_k^n\rangle = \frac{1}{\sqrt{d}} \sum_{j=a}^{d} e^{(2\pi ijk/d)}|\underline{nj}\rangle$$

and each of the split chains is indexed from a to d; i.e., the extremal nodes are $1a$ to $1d$ and Ma to Md (due to symmetry, it is entirely irrelevant which are labeled as which). By definition,

$$e^{-iH_{\text{star}}t_0}|W_0^n\rangle = |W_0^{M+1-n}\rangle.$$

To derive the dynamics of the other states $|W_k^n\rangle$, note that H_{star} decomposes into a further direct sum structure of fixed k, each with a hopping Hamiltonian which is exactly the same effective Hamiltonian as for H_{chain} acting on states of the form $(|\underline{m}\rangle - |\underline{M+1-m}\rangle)/\sqrt{2}$. Consequently, for $k \neq 0$,

$$e^{-iH_{\text{star}}t_0}|W_k^n\rangle = -|W_k^n\rangle.$$

By restricting to these intervals of t_0, we find a very simple description of the unitary evolution of states on the star subsystem, which we can make use of for designing the routing protocols within the subsystem.

Routing within a subsystem.—Our aim is now to route an input state within a subsystem, i.e., to transmit it from spin $1j$ to $1l$. The input and output states can be written as $|\underline{1j}\rangle = \frac{1}{\sqrt{d}}\sum_k e^{-(2\pi ijk/d)}|W_k^1\rangle$, which differ only by the relative phases of the $|W_k^1\rangle$ states. We start manipulating the phases by first evolving for time t_0, applying a phase gate of phase θ, $Z^{(\theta)}$, on each of the spins $1j$, and waiting another transfer time t_0. The relative phase of the $|W_0^1\rangle$ component is shifted:

$$\left(\sum_{k\neq 0} e^{(-2\pi ijk/d)}|W_k^1\rangle + e^{-i\theta}|W_0^1\rangle\right)/\sqrt{d}.$$

Now all we have to do is apply local phase gates on each spin $1j$ with the cumulative effect that

$$\prod_{j=a}^{d} Z_j^{(2\pi j/d)}|W_k\rangle = |W_{k+1 \bmod d}\rangle,$$

enabling us to permute through each $|W_k\rangle$ and alter its phase, creating the state we need after only time $2dt_0$. Hence, we can route from any node $1j$ to any other $1l$.

In fact, one can prove that the magnetic fields give us full control over the single excitation subspace of a subsystem, so we can create any single excitation state that we want across the input and output nodes, such as an entangled state, which can subsequently be distributed using the state transfer protocol. If d is even, the construction is particularly simple because then we can use the phase changing protocol to create

$$\frac{1}{\sqrt{d}}\sum_j e^{i\pi(-1)^j/4}|W_j\rangle = \frac{1}{\sqrt{2}}\left(|\underline{1d}\rangle + i\left|\underline{1\tfrac{d}{2}}\right\rangle\right)$$

from an excitation initially localized on a single site.

Our construction is general enough to take advantage of any one of the infinitely many solutions to the perfect state transfer problem using a fixed Hamiltonian [4,7], but there is a particularly promising choice if $d = 3$ (i.e., 2D triangular or cubic lattice). Here we select the standard perfect state transfer solution for $M = 5$ [2], i.e., $K_1 = \sqrt{2}$ and $K_2 = \sqrt{3}$, meaning that the regular network that we construct has every coupling strength taking on the same modulus. Furthermore, this solution is the most efficient solution to the state transfer problem with regard to a number of parameters [5,7,22].

Implementation.—The techniques described here can be applied to any particle number preserving local Hamiltonian, and therefore in principle to a vast range of experiments. One of the most natural candidates is an optical lattice, in which vast arrays of atoms can be prepared in a particular hyperfine ground state (corresponding to logical $|0\rangle$ of a qubit) in periodically defined lattices [23] with local Hamiltonian interactions [24] where controls can be applied periodically [25]. The intersite couplings in the limit $d_{nm} \ll U_n$, U_m take the form $J_{n,m} = d_{nm}^2(1/U_n + 1/U_m)$, where the d_{nm} are tunable for each edge in a subsystem via the depth of the trapping potential, and the U_n are collisional coupling strengths on site n. These collisional couplings can be tuned through a vast

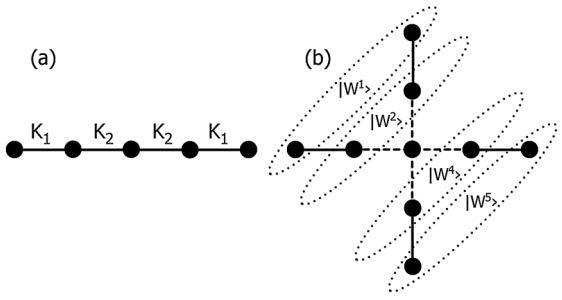

FIG. 3. (a) A perfect state transfer coupling scheme for a chain of 5 qubits. (b) Conversion of (a) into a star topology ($d = 2$), with $|\underline{n}\rangle$ in (a) transforming into $|W_0^n\rangle$. Dashed coupling strengths are scaled to $K_2/\sqrt{d}$ for $2d$ branches.

range of positive and negative values using Feshbach resonances, enabling the required patterning of coupling strengths. In any such experiment, it would be preferable that the local magnetic field controls can be applied slowly rather than quickly. Given that we have shown there is at least one efficient solution to the state routing problem with the given controls, it is a simple problem to numerically find other pulse sequences under constraints such as pulse strength and rate of change.

Conclusions.—In this Letter, we have argued that routing will be much more useful in a computational architecture than state transfer. We have shown how regular networks which are local in any number of spatial dimensions d can be designed to route quantum states between arbitrary nodes in a time that is linear in the distance to be covered. Multiple states can be transferred at once, meaning that a transfer rate can be realized which is in excess of that achievable in perfect state transfer schemes [5,7], even in the 1D case. Achieving this required a level of control that was never explicitly utilized in previous perfect state transfer schemes until its recent observation in [19], but was implicitly present for the addition and removal of states. These controls can be applied in a global way by addressing, for instance, every fifth spin at regular intervals. These properties therefore bring such a scheme much closer to reality, and we have presented optical lattices as a natural candidate.

Before such experiments can be seriously considered, a study of errors would be necessary. We defer this to future studies, but conclude with some strong motivation for the error tolerance of this model. If manufacturing errors can be identified on a given subsystem, then provided these errors are not on the input or output blocks, they can be routed around. Let us assume that a manufacturing error occurs within a subsystem with a probability p, but it is promised to preserve the symmetry properties in the V structures [26]. On detecting an error, we route around this particular subsystem for transmission. Considering the underlying lattice (where we treat the overall lattice as being composed of a convolution between the underlying lattice and the subsystem), this corresponds to removing a single site with probability p. From percolation theory, long-range connectivity (and hence communication) remains provided $1 - p > p_c$, the site percolation threshold of the underlying lattice. We can choose the underlying geometry to optimize this tolerance using, for instance, a triangular lattice in 2D, since it has a smaller percolation threshold ($p_c = \frac{1}{2}$) than a square lattice. Future analysis may include a treatment of cooperating parties—it is currently assumed that parties at all sites cooperate, but how many error prone participants can be tolerated? Can this be protected against through the use of error correcting codes or Byzantine agreement protocols? If uncooperative parties can be identified, then they can be routed around in much the same way as faults in the manufacture of the network.

A unique feature of one-dimensional state transfer systems is their ability to transfer a state irrespective of how the rest of the chain is initialized [21,27]. Recovering this feature in routing scenarios is a significant challenge. However, the subsystem structure means that we only need to correctly prepare the state of a small number of subsystems, and the magnetic field controls prove very helpful in doing so. Indeed, by only initializing subsystems as needed, we can remove any noise that has occurred on these subsystems while routing is being performed elsewhere in the network, where state transfer requires the entire chain to remain noise free.

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