Local Diffusion Theory for Localized Waves in Open Media

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We report a first-principles study of static transport of localized waves in quasi-one-dimensional open media. We find that such transport, dominated by disorder-induced resonant transmissions, displays novel diffusive behavior. Our analytical predictions are entirely confirmed by numerical simulations. We show that the prevailing self-consistent localization theory [B. A. van Tiggelen *et al.*, Phys. Rev. Lett. **84**, 4333 (2000)] is valid only if disorder-induced resonant transmissions are negligible. Our findings open a new direction in the study of Anderson localization in open media.

Introduction.—In the past years experimental studies of localization have been boosted due to the unprecedented level of manipulating ultracold atomic gases [1], dielectric materials [2–7], and elastic media [8]. A key feature shared by many experimental setups [5–9] is that, to facilitate measurements, wave energies leak out of systems through boundaries. Consequently, wave interferences interplay strongly with the wave energy leakage that, conceptually, enriches transport phenomena of localized waves while, technically, it pushes forward developments of theoretical approaches. In particular, as a unique property of finite-sized samples, localized states in the sample center create resonant transmissions [10,11]. Although these transmissions are rare events, nevertheless, they contribute significantly to average transmission. In fact, random matrix theory predicts that in quasi-one-dimension (quasi-1D) the localization length measured from average transmission can be 4 times larger than that measured from typical transmission [12]. Recently, disorder-induced resonant transmissions have found considerable practical applications. For example, they mimic a "resonator" with high-quality factors and thus are used to fabricate random laser [13] and to realize optical bistability [14].

However, to study this intriguing interplay has proved to be, in general, a formidable task. Confronting this challenge, a decade ago van Tiggelen, Lagendijk, and Wiersma hypothesized [15] a so-called self-consistent local diffusion (SCLD) model for localization in open media (for a review, see Ref. [16]). They phenomenologically generalized the self-consistent localization theory of infinite media [16] by demanding the diffusion coefficient to be position dependent so as to take boundary effects into account. The SCLD model (as well as its dynamic generalization [8,17]), having the advantages of physical transparency and methodological simplicity over other approaches, is guiding considerable experimental and theoretical activities (e.g., Refs. [7,8,18]). Yet, the validity of the SCLD model inside localized samples is largely unknown and, in fact, has been severely questioned by recent pulsed

microwave experiments [7]. There it was shown that the (dynamic) SCLD model [17] fails to describe transport in quasi-1D localized samples at long times, when energies are mainly stored in long-lived modes. The dramatic discrepancy [7] between experimental measurements and theoretical predictions is conveying an opinion. That is, the highly nonlocal object of disorder-induced resonant transmission [10], which plays a decisive role in transport of localized waves [11], may not be captured by such a model.

Motivated by these activities, we performed a first-principles study of static wave transport in quasi-1D localized samples, i.e., $L \gg \xi$ with L and ξ the sample and localization length, respectively. We predicted analytically and confirmed numerically that, in these systems, localized waves display a novel diffusion phenomenon. Our theory shows that the SCLD model is valid only if disorder-induced resonant transmissions are negligible. Our findings, capable of being generalized to higher dimensions, may open a new direction in the study of Anderson localization in open media.

Main results and qualitative discussions.—We placed a source at x' and considered the average wave intensity at x (x, x' are measured from given boundary), denoted as $\mathcal{Y}(x, x')$. Our first-principles analytic theory, justifying the static local diffusion equation, $-\partial_x D(x) \partial_x \mathcal{Y}(x, x') = \delta(x - x')$, leads to the following central results. (i) The local (or position-dependent) diffusion coefficient $D(x)$ displays a novel scaling behavior. Specifically, $D(x)$ depends on x via the scaling $\lambda = (L - x)x/(L\xi)$ [$D_0 = D(0)$],

$$D(x)/D_0 = D_\infty(\lambda), \qquad (1)$$

and the scaling function $D_\infty(\lambda)$ is $\sim e^{-\lambda}$ for $\lambda \to \infty$. (ii) From (i) it follows that inside the sample, surprisingly, $D(x)$ is enhanced drastically from the exponential decay,

$$D(x)/D_0 \propto e^{x^2/(L\xi)} e^{-x/\xi}, \qquad \xi \ll x \leq L/2 \quad (2)$$

[$D(x) = D(L - x)$]. (iii) Equations (1) and (2) are universal regardless of the time-reversal symmetry. Our results, while

entirely confirmed by simulations (Fig. 1), show that the
SCLD model fails in localized samples.

Let us first present qualitative explanations of the main
results (i)–(iii). The scaling behavior, Eq. (1), finds its origin
in wave interferences. Indeed, as waves penetrate into a
time-reversal medium, they may counterpropagate along
the same loop and interfere with each other—the well-
known weak localization [19]. However, different from
infinite media, in open media, wave energies leak out
through the boundaries and, as such, the probability (in the
frequency domain) of forming a loop is finite in the static
limit, which is $\propto (L - x)x/L$ in quasi-1D. Then, a one-loop
wave interference correction to D_0 results, which is x
dependent and of the order of $\lambda = (L - x)x/(L\xi)$.
Furthermore, because λ monotonically decreases from the
sample midpoint to boundaries, as waves propagate towards
the sample center, the propagation paths tend to form more
(say, n) loops with a probability $\propto [(L - x)x/L]^n$, leading
to a wave interference correction $\sim \lambda^n$. Thus, wave interfer-
ences everywhere render $D(x)$ depending on x via λ.

That the scaling behavior of local diffusion is far beyond
the reach of the SCLD model is best appreciated by broken
time-reversal systems (unitary symmetry). There, the above
one-loop wave interference is absent and, therefore, the
SCLD model ceases to work. [Indeed, for $\lambda \to 0$, the linear
term in the λ expansion of $D_\infty(\lambda)$ now disappears.] Instead,
two paths may take the same route, propagate in the same
direction, but visit individual scatterers at different times. As
such, they form loops and equally contribute wave interfer-
ence corrections to D_0. The perturbative λ expansion is
thereby justified, with the leading order correction $\sim \lambda^2$.

Having explained Eq. (1), let us estimate the asymptotic
form of $D_\infty(\lambda)$ at $\lambda \to \infty$. To this end we enjoy the
universality of $D_\infty(\lambda)$ and set $L \to \infty$ (semi-infinite
media). In this simple case, on physical grounds, we expect
$D(x)/D_0 = D_\infty(\lambda) \sim e^{-x/\xi}$ for $x \gg \xi$. Because of

FIG. 1 (color online). Comparing results obtained from nu-
merical simulations, the analytical prediction (2) (solid lines)
and the SCLD model (dashed lines). $D(x)/D_0$ was computed
numerically for two wave frequencies, $\omega = 1.65c/a$ (squares)
and $\omega = 0.72c/a$ (circles), and for five different sample lengths,
$L/\xi = 2.5, 5, 10, 15,$ and 20.

$\lambda = x/\xi$, we find $D_\infty(\lambda) \sim e^{-\lambda}$ for $\lambda \gg 1$. This asymp-
totic form then gives Eq. (2) for finite-sized samples.
Importantly, the significant enhancement from the expo-
nential decay is related to the fluctuation of the inverse
localization length γ in finite-sized samples. Indeed, for
$L \gg \xi$, the distribution of γ is Gaussian, with the average
and variance being ξ^{-1} and $2/(\xi L)$, respectively [12,20].
Averaging $e^{-\gamma x}$, we obtained

$$\int_0^\infty d\gamma e^{-\gamma x} e^{-(\xi L/4)(\gamma - \xi^{-1})^2} \sim D(x) \qquad (3)$$

for x deep inside the sample.

Failure of SCLD model.—For simplicity we focus on
classical scalar waves, and begin with testing the validity
of the SCLD model [15]. We performed numerical simula-
tions of the spatially resolved wave intensity across a ran-
domly layered medium, which is embedded in an air
background and excited by a plane wave of (angular) fre-
quency ω. The layer thickness is a, and the relative permit-
tivity at each layer fluctuates independently, with a uniform
distribution in the interval $[1 - \sigma, 1 + \sigma]$. Here σ measures
the degree of randomness of the system, and throughout
this work we considered nonreflecting boundaries. We
set $\sigma = 0.7$ and considered two wave frequencies, $\omega =
1.65c/a$ and $0.72c/a$, where c is the speed of light in air.
We used the standard transfer matrix method to calculate the
transmission coefficient T_β and wave intensity distribution
$I_\beta(x)$ for each configuration β. For each ω, we calculated
the ensemble-averaged current $j \equiv \langle T_\beta \rangle$ and wave intensity
distribution $I(x) \equiv \langle I_\beta(x) \rangle$ of $2\,000\,000$ realizations of di-
electric disorders for different sample lengths. Since the
current across the sample is uniform due to the conservation
law, we used the relation $j = -D(x)\partial_x I(x)$ to compute $D(x)$
by presuming the static local diffusion equation.

In Fig. 1 the results of $D(x)/D_0$ obtained by simulations
and by numerically solving the SCLD model are presented.
First of all, they both show that $D(x)$ tends to decay
exponentially from the boundary in the limit $L \to \infty$ (dot-
ted line). The decay length (the localization length) ξ
was found to be the transport mean free path, which is
$21a$ ($50a$) for $\omega = 1.65c/a$ ($0.72c/a$). We therefore re-
scale x into x/ξ and present results for five different
sample lengths, $L/\xi = 2.5, 5, 10, 15,$ and 20. We see
that for different frequencies the simulation results
(squares and circles) overlap, signaling the scaling behav-
ior independent of the parameters of random media. It is
obvious that, except near the boundaries, the results from
simulation are significantly larger than those from the
SCLD model (dashed lines). The deviation is prominent
for large L/ξ, where the results from the SCLD model
converge to the sum of two truncated exponentials, decay-
ing from their respective boundaries. Does localized waves
in open media display diffusive transport? Our analytical
prediction below provides a definitive answer to this con-
ceptually important question. In particular, the analytical
result of $D(x)/D_0$, Eq. (2), is in excellent agreement with
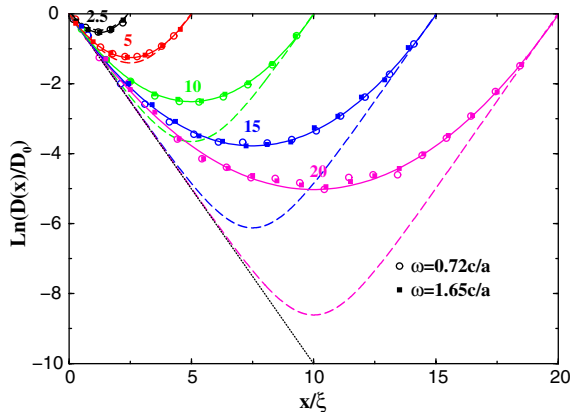numerical simulations (solid lines).

Exact microscopic formalism.—Referring to a separate publication for technical details, we turn now to outline the proof, and the strategy is as follows. In the framework of the supersymmetric field theory of localization [21,22], we calculated explicitly $\mathcal{Y}(x, x')$, and found that it solves the local diffusion equation. In doing so, we managed to calculate the weak localization correction, $\delta D(x)$. Then, we found the Gell-Mann–Low equation of the local diffusion coefficient, $D(x) = D_0 + \delta D(x)$, which eventually leads to Eq. (2).

In the present context the supersymmetric technique has many advantages over others. The key ingredient is the introduction of a "spin" Q (to be defined below) that encapsulates wave interferences by fluctuations of the "spin direction": the larger the fluctuation, the stronger the localization effect. With the help of the Q spin, the picture of localization in open media is analogous to that of the more familiar problem—finite classical ferromagnetic spin chains (but formal treatments are not). In the latter system, the two end spins are fixed and parallel, and the spin direction fluctuates elsewhere with a small (large) fluctuation amplitude closed to the chain ends (midpoint). Translated to the Q-spin language, such inhomogeneous fluctuations reflect the spatial inhomogeneity of wave interferences in open media—the very mechanism of local diffusion. Most importantly, the feature that the two Q spins at the boundaries are "parallel" takes all disorder-induced resonant transmissions into full account. Thus, the phenomenology of local diffusion is substantiated by a completely microscopic formalism, albeit in an elegant manner.

Formally, the spin Q is an 8×8 supermatrix defined on the advanced-retarded (AR), bosonic-fermionic (BF), and time-reversal (TR) sector. The "AR" sector accommodates different analytic structures of the advanced (retarded) Green function. The "BF" sector accommodates the supersymmetry: the diagonal (off-diagonal) matrix elements are commuting (anticommuting) numbers. The "TR" sector accommodates the time-reversal symmetry. The leakage enters through the boundary constraint $Q(0) = Q(L) = \sigma_3^{AR}$. Here, $\sigma_3^X = \text{diag}(1, -1)$, $X = \text{AR}$, BF, TR. Then, $\mathcal{Y}(x, x')$ is exactly expressed as

$$\mathcal{Y}(x,x') = \frac{\pi\nu}{2^7} \int D[Q] e^{-F[Q]} \text{str}[\sigma_3^{BF}(1 + \sigma_3^{AR})$$
$$\times (1 - \sigma_3^{TR})Q(x)(1 - \sigma_3^{AR})(1 - \sigma_3^{TR})\sigma_3^{BF}Q(x')].$$
$$(4)$$

The action $F[Q] = -\frac{\pi\nu D_0}{8} \int_0^L dx\, \text{str}(\partial_x Q)^2$ (ν the density of states per unit length and "str" the supertrace) is the energy cost of the Q-field fluctuations. It introduces a characteristic scale—the localization length $\xi \propto \pi\nu D_0$.

Importantly, the mean field $Q(x) = \sigma_3^{AR}$, compatible with the boundary constraint, minimizes the action. Bearing this in mind, we introduced the parametrization $Q = (1 + iW)\sigma_3^{AR}(1 + iW)^{-1}$, where $W(x)$ anticommutes with σ_3^{AR} and vanishes at $x = 0, L$, and performed the W expansion. By keeping the leading order W expansions, we

obtained from Eq. (4) bare $\mathcal{Y}(x, x')$, which solves $-D_0\partial_x^2\mathcal{Y}(x, x') = \delta(x - x')$ with the boundary condition $\mathcal{Y}(0, x') = \mathcal{Y}(L, x') = 0$. Calculating the leading wave interference corrections to $\mathcal{Y}(x, x')$, we found

$$\mathcal{Y}(x,x') = \mathcal{Y}(x,x') - \int dy\, \mathcal{Y}(x,y)\partial_y\delta D(y)\partial_y \mathcal{Y}(y,x'), \quad (5)$$

with the leading order weak localization correction

$$\frac{\delta D(x)}{D_0} = \alpha\frac{\mathcal{Y}(x, x)}{\pi\nu} - \frac{1}{2}(1 - \alpha^2)\left[\frac{\mathcal{Y}(x, x)}{\pi\nu}\right]^2, \quad (6)$$

where $\mathcal{Y}(x, x) = (L - x)x/(D_0 L)$, and $\alpha = -1$ for the orthogonal symmetry while $\alpha = 0$ for the unitary symmetry. Equation (6) justifies that $D(x)$ depends on x via λ.

For the unitary symmetry the first term of Eq. (6) vanishes, reflecting that the one-loop interference is infeasible. Thus, the local diffusion and scaling behavior of $D(x)$ are universal concepts, extrinsic to the time-reversal symmetry that is required by the SCLD model.

Scaling theory of local diffusion and disorder-induced resonant transmission.—We now make an important observation: $\mathcal{G}(\lambda) = \nu D(x)/(\xi\lambda)$ and λ, formally, play the role of the "Thouless conductance" [19] and the "system size," respectively. Indeed, from Eq. (6) we found

$$\frac{d\ln\mathcal{G}}{d\ln\lambda} = \beta(\mathcal{G}) = -1 + c_1\mathcal{G}^{-1} + c_2\mathcal{G}^{-2} + \cdots \mathcal{G} \gg 1,$$
$$(7)$$

with $c_1 < 0$ for $\alpha = -1$ and $c_1 = 0$, $c_2 < 0$ for $\alpha = 0$. This is fully analogous to the usual one-parameter scaling theory of quasi-1D localization [19], where, in particular, the perturbative expansion of the β function also finds its origin in weak localization [16,22].

From Eq. (6), we further found that the weak localization corrections of open and infinite media, including the coefficients, are identical except that the returning probability, $\mathcal{Y}(x, x)$, replaces that of infinite media. This duality persists in all the higher-order weak localization corrections, and, as such, the β function here is identical to that of the usual one-parameter scaling theory [19]. Identifying this duality, we followed Refs. [19,20] to extrapolate Eq. (7) into the regime of $\mathcal{G} \ll 1$, obtaining

$$\beta(\mathcal{G}) = \ln\mathcal{G}, \qquad \mathcal{G} \ll 1. \quad (8)$$

The scaling theory of local diffusion, namely, Eqs. (7) and (8), is far beyond the reach of earlier theoretical studies [21,23] and has far-reaching consequences. (It may be reproduced within the formalism of Ref. [24].) In particular, Eq. (8) gives $\mathcal{G}(\lambda) \propto e^{-\lambda}$ for $\lambda \to \infty$ and thus Eq. (2) which, as shown below, fully captures the rare disorder-induced resonant transmission. Solving the static local diffusion equation, we found that the ensemble-averaged transmission is $\langle T(L)\rangle \propto [\int_0^L dx/D(x)]^{-1}$ [This is regardless of the explicit form of $D(x)$, as first noticed in Ref. [15].] Inserting Eq. (2) into it gives $\langle T(L)\rangle \propto e^{-L/(4\xi)}$ for $L \gg \xi$. On the other hand, noticing that $T(L) = e^{-\gamma L}$,
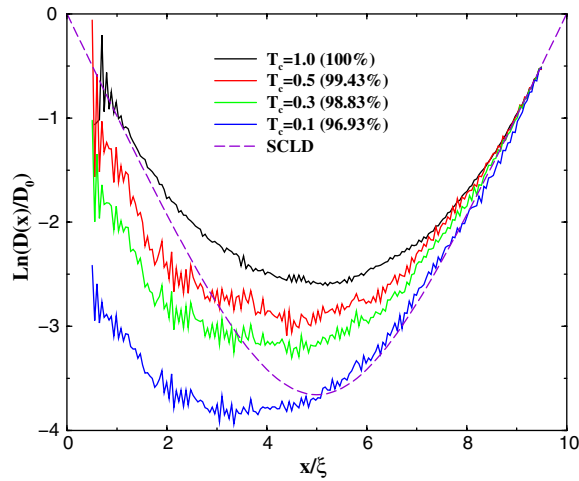
FIG. 2 (color online). Simulation results obtained by eliminating the high-transmission ($T > T_c$) states in the original ensemble ($\omega = 1.65c/a$ and $L/\xi = 10$). The source is placed at $x = 0$.

we found that the typical transmission gives $d\langle \ln T(L)\rangle/dL = -\xi^{-1}$ from the Gaussian distribution of γ [cf. Eq. (3)]. Thus, the localization lengths obtained by the arithmetic and geometric means differ by a factor of 4 irrespective of orthogonal or unitary symmetry. This is in agreement with the result of the random matrix theory, and is because $\langle T(L)\rangle$ is dominated by disorder-induced resonant transmissions [12].

To further study effects of rare high-transmission states, we analyzed all the samples in Fig. 1 ($\omega = 1.65c/a$). First of all, the distribution of $\ln T$ is well fitted by the normal distribution, with the average $\approx -L/\xi$ and the variance $\approx 2L/\xi$. This confirms the Gaussian distribution of γ. Then, we intentionally eliminated a small fraction of high-transmission ($T > T_c$) states (composed mostly of singly localized states and of a small portion of necklace states [25]) from the original ensemble ($L/\xi = 10$) and recomputed $D(x)/D_0$. As shown in Fig. 2, strikingly, even when the fraction of removed states is as small as 0.6% (red solid line), the simulation result deviates drastically from the original one (black solid line) and is asymmetric, signaling the breakdown of local diffusion. Thus, we found that rare high-transmission states are essential to establish local diffusion and scaling behavior.

Conclusions.—We found in quasi-1D localized samples a scaling behavior of the (static) local diffusion coefficient capturing all the rare disorder-induced resonant transmission. Our findings show unambiguously that the prevailing SCLD model is valid only if rare disorder-induced resonant transmissions are negligible which, nevertheless, play a decisive role in transport of localized waves.

The found phenomenon is intrinsic to finite-sized samples with open boundaries and does not exist in an infinite sample. It is an unconventional diffusion phenomenon in the sense that the diffusion coefficient can drop by many orders of magnitude as the position changes from the boundary to the midpoint. (For diffusive samples, such a position dependence is weak and thus does not lead to any interesting phenomenon other than ordinary diffusion.) It is such a drastic change (in the diffusion coefficient) that leads to a global localization behavior as shown in the scaling of the average transmission which decays exponentially with sample size.

Our theory has many immediate applications. For example, it can be directly used to study the speckle pattern of scattered waves which has recently attracted considerable attentions. It may also be generalized to higher dimensions for studying disorder-induced resonant transmissions in (or close to) the localization regime. This issue has practical applications such as random laser.

Note added in proof.—After this Letter was submitted, a related work was published by Payne *et al.* [26].

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