Test Bodies and Naked Singularities: Is the Self-Force the Cosmic Censor?

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Jacobson and Sotiriou showed that rotating black holes could be spun up past the extremal limit by the capture of nonspinning test bodies, if one neglects radiative and self-force effects. This would represent a violation of the cosmic censorship conjecture in four-dimensional, asymptotically flat spacetimes. We show that for some of the trajectories giving rise to naked singularities, radiative effects can be neglected. However, for these orbits the conservative self-force is important, and seems to have the right sign to prevent the formation of naked singularities.

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The most general stationary vacuum black-hole (BH) solution of Einstein's equations in a four-dimensional, asymptotically flat spacetime is the Kerr geometry [1], characterized only by its mass M and angular-momentum J. Solutions spinning below the Kerr bound $cJ/GM^2 \leq 1$ possess an event horizon and are known as Kerr BHs. Solutions spinning faster than the Kerr bound describe a "naked singularity," where classical general relativity breaks down and (unknown) quantum gravity effects take over. It was hypothesized by Penrose that classical general relativity encodes in its equations a mechanism to save it from the breakdown of predictability. This is known as the cosmic censorship conjecture (CCC) [2], which asserts that every singularity is cloaked behind an event horizon, from which no information can escape.

There is no proof of the CCC. Indeed there are a few known counter examples, but these require either extreme fine-tuning in the initial conditions or unphysical equations of state [2], or are staged in higher-dimensional spacetimes [3]. Moreover, all existing evidence indicates that Kerr BHs are perturbatively stable [4], while Kerr solutions with $cJ/GM^2 > 1$ are unstable [5]. Thus, naked singularities cannot form from BH instabilities.

Because naked singularities appear when $cJ/GM^2 > 1$, it is conceivably possible to form them by throwing matter with sufficiently large angular momentum into a BH. With numerical-relativity simulations, the authors of Ref. [6] found no evidence of a formation of naked singularities in a high-energy collision between two comparable-mass BHs: either the full nonlinear equations make the system radiate enough angular momentum to form a single BH, or the BHs simply scatter. The case of a test particle plunging into an extremal Kerr BH was studied by Wald [7], who showed that naked singularities can never be produced, because particles carrying dangerously large angular momentum are just not captured.

Recently, Jacobson and Sotiriou (JS) [8] (building on Refs. [9]) have shown that if one considers an almost extremal BH, nonspinning particles carrying enough angular momentum to create naked singularities are allowed to be captured [10]. As acknowledged by JS, however, their analysis neglects the conservative and dissipative self-force (SF), and both effects may be important [11]. In this Letter we will show that the dissipative SF (equivalent to radiation reaction, i.e., the energy and angular momentum losses through gravitational waves) can prevent the formation of naked singularities only for some of JS' orbits. However, we will show that for all these orbits the conservative SF is comparable to the terms giving rise to naked singularities, and should therefore be taken into account. Hereafter we set $G = c = M = 1$.

JS considered a BH with spin $a \equiv J/M^2 = 1 - 2\epsilon^2$, with $\epsilon \ll 1$, and a nonspinning test particle with energy E, angular momentum L, and mass m. Neglecting the dissipative and conservative SF, the particle moves on a geodesic, and JS identified a class of equatorial geodesic orbits such that (i) the particle falls into the BH, which implies an upper limit on the angular momentum, $L < L_{\rm max}$, and (ii) the BH is spun up past the extremal limit and destroyed, which implies a lower limit on the angular momentum, $L > L_{\rm min}$. Therefore,

$$L_{\rm min} = 2\epsilon^2 + 2E + E^2 < L < L_{\rm max} = (2 + 4\epsilon)E. \quad (1)$$

Imposing $L_{\rm max} > L_{\rm min}$ then yields

$$E_{\rm min} = (2 - \sqrt{2})\epsilon < E < E_{\rm max} = (2 + \sqrt{2})\epsilon. \quad (2)$$

Finally, JS checked that these intervals contain both bound orbits (i.e., orbits that start with zero radial velocity at finite radius) and unbound orbits (i.e., orbits that start from infinity). Parametrizing the above interval as

$$E = E_{\rm min} + x(E_{\rm max} - E_{\rm min}) = E_{\rm min} + 2x\sqrt{2}\epsilon \quad (3)$$

$$L = L_{\min} + y(L_{\max} - L_{\min}) = L_{\min} + 8y\epsilon^2(1-x)x \quad (4)$$

with $0 < x < 1$, $0 < y < 1$, the final spin is

$$a_f^{\rm JS} = \frac{a+L}{(1+E)^2} = 1 + 8\epsilon^2(1-x)xy + \mathcal{O}(\epsilon^3) > 1, \quad (5)$$

and the spin up is due to the terms quadratic in ϵ.

Let us first investigate how radiation reaction affects JS' analysis. Taking radiation losses $E_{\rm rad}$ and $L_{\rm rad}$ into account, Eq. (5) becomes

$$a_f = 1 + 8\epsilon^2(1-x)xy + 2E_{\rm rad} - L_{\rm rad} + \mathcal{O}(\epsilon^3). \quad (6)$$

Let us focus on unbound geodesics [12], and following JS assume $E/m \gg 1$ and $L/m \gg 1$ (null orbits). These orbits are characterized by the impact parameter $b = L/E$ alone. From Eqs. (1) and (2), JS's orbits have $L = bE$, with $b = 2 + 4\epsilon\{1 - 2x(x-1)(y-1)/[2+\sqrt{2}(2x-1)]\}$. Varying x and y between 0 and 1, one obtains $b = 2 + \delta\epsilon$, with $2\sqrt{2} < \delta < 4$. However, because $b_{\rm ph} = 2 + 2\sqrt{3}\epsilon + \mathcal{O}(\epsilon^2)$ is the impact parameter of the circular photon orbit ("light ring"), only orbits with $2\sqrt{2} < \delta < 2\sqrt{3}$ are unbound. When $\delta \approx 2\sqrt{3}$, these orbits are expected to circle many times around the light ring, so the radiation reaction could prevent the formation of naked singularities or at least invalidate JS' analysis. In fact, for δ arbitrarily close to $2\sqrt{3}$, the particles would orbit around the light ring an arbitrarily large number of times, and gravitational-wave emission must be important [13]. We will show, however, that this is not true for all of JS' orbits.

Considering the geodesic equations for null equatorial orbits with impact parameter $b = b_{\rm ph}(1-k)$, with $k \ll \epsilon \ll 1$, one finds that the radial potential—defined as $V_r(r) \equiv (dr/d\lambda)^2$ with λ an affine parameter—has a minimum at $r = r_{\min} = r_{\rm ph} + \mathcal{O}(k)$, near which

$$\frac{d\phi}{dr} \approx \left(\frac{8}{3} + \frac{\sqrt{3}}{2\epsilon}\right)\left[\frac{8}{\sqrt{3}}k\epsilon + 3(r - r_{\min})^2\right]^{-1/2}. \quad (7)$$

Integrating from $r_{\min} - \Delta r_2$ to $r_{\min} + \Delta r_1$, with $\Delta r_{1,2} \gg k\epsilon$, the number of cycles near the minimum is

$$N_{\rm cycles} \approx \int_{r_{\min}-\Delta r_2}^{r_{\min}+\Delta r_1} \frac{d\phi}{dr}\frac{dr}{2\pi} = [A + B\log(k\epsilon)]\left(\frac{8}{3} + \frac{\sqrt{3}}{2\epsilon}\right) \quad (8)$$

A and B being constants depending on the integration interval. Fixing ϵ, and thus the BH spin, we can see that $N_{\rm cycles}$ depends on $\log k$, and diverges when $k \to 0$ [13].

Because the fluxes are proportional to $N_{\rm cycles}$, we have

$$E_{\rm rad} = \Delta E(\epsilon) \times N_{\rm cycles}, \qquad L_{\rm rad} = \Delta L(\epsilon) \times N_{\rm cycles}, \quad (9)$$

where ΔE and ΔL are the fluxes in a single orbit. From a frequency-domain analysis [14], $\Delta E/\Delta L$ must equal the light-ring frequency, $\Omega_{\rm ph} \approx 1/2 - (\sqrt{3}/2)\epsilon$; hence,

$$\Delta E(\epsilon) = E_1(\epsilon)(1 + e_2\epsilon), \quad (10)$$

$$\Delta L(\epsilon) = 2E_1(\epsilon)[1 + (\sqrt{3} + e_2)\epsilon]. \quad (11)$$

Here $E_1(\epsilon)$ is the energy flux for a single orbit at leading order in ϵ, and e_2 is an undetermined coefficient. Semiquantitative arguments by Chrzanowski [15] and more rigorous analytical calculations by Chrzanowski and Misner [16] show that,

$$E_1 \sim (r - r_H)E^2 \sim \epsilon E^2 \sim \epsilon^3 \quad (12)$$

(later we will discuss an additional proof of this scaling). At leading order in ϵ, this results in

$$E_{\rm rad} = \Delta E(\epsilon) \times N_{\rm cycles} \sim \log(k\epsilon)\epsilon^2. \quad (13)$$

This scaling still depends on k, but the dependence is logarithmic, so unless k is really small $E_{\rm rad} \sim \log(\epsilon)\epsilon^2$. Although terms of order $\epsilon^2 \log\epsilon$ seem to dominate Eq. (6), because of Eqs. (9)–(11) one has $L_{\rm rad} - 2E_{\rm rad} = 2\sqrt{3}\epsilon E_1(\epsilon)N_{\rm cycles} \sim \epsilon^3 \log\epsilon$. Therefore JS' analysis is valid for these trajectories. However, if $k \lesssim \exp(-1/\epsilon)$, $E_{\rm rad} \sim \epsilon$ and $L_{\rm rad} - 2E_{\rm rad} \sim \epsilon^2$, and JS' analysis is not valid because radiative effects cannot be neglected.

To test the above picture we used a time-domain code [17] solving the inhomogeneous Teukolsky equation [14] that describes the gravitational perturbations of Kerr BHs in the context of extreme mass-ratio binaries. This code has been successfully used in many scenarios, including an extensive study of recoil velocities from extreme mass-ratio binaries [18]. Because, for almost extremal BHs and in Boyer-Lindquist coordinates, the particle's orbit, the light ring, and the horizon are extremely close, we modeled the test particle to have a fixed width in the "tortoise" coordinate r^* as opposed to r [19], and checked that our results are independent of the particle's width when that is sufficiently small. (More details on these tests will be presented in a follow-up paper.)

We consider BHs with $a = 0.99$, 0.992, 0.994, 0.996, 0.998, and 0.999 and geodesics having $E = (E_{\max} + E_{\min})/2 = 2\epsilon$, $L = b_{\rm ph}E(1-k)$ with $k = 10^{-5}$, and $m = 0.001 \ll E$. Using these geodesics and integrating their cycles from $r = 1.05r_{\rm ph}$ to $r = (r_{\rm ph} + r_{\rm hor})/2$ ($r_{\rm ph}$ being the light-ring radius), we get $A \approx 0.3294$, $B \approx -0.01941$ for the coefficients in Eq. (8). Assuming $E_1 = e_1\epsilon^n$, we fit the energy and angular-momentum fluxes at infinity with Eqs. (9)–(11), obtaining $n \approx 2.91$. Because this is very close to the theoretical value $n = 3$, we assume $n = 3$ and fit the data with only two free parameters, e_1 and e_2, obtaining $e_1 = 136.97$ and $e_2 = -4.423$. With these values, Eqs. (9)–(11) reproduce the numerical data to within 1%–3% for $a < 0.999$, which is comparable to the data accuracy. For $a = 0.999$, however, the fluxes predicted by Eqs. (9)–(11) are about 12% larger than the numerical ones. To investigate this issue, we ran an additional simulation for $a = 0.9998$, which seems to confirm that Eqs. (9)–(11) overpredict the fluxes for very high spins. At this stage it is not clear whether this is a numerical problem (simulations are very challenging for $a \approx 1$) or whether this is due to the simplified analytical derivation of Eqs. (9)–(11). We will investigate this issue in the follow-up paper, but because the numerical fluxes are *smaller* than expected, it only reinforces our conclusion that there are orbits giving rise to naked singularities *even* when radiation reaction is taken into account.

Since $L_{\rm rad} - 2E_{\rm rad} = 2\sqrt{3}\epsilon E_1(\epsilon) N_{\rm cycles} \sim \epsilon^3 \log(k\epsilon) > 0$, Eq. (6) predicts that radiation reaction will decrease the final spin a_f. Using the above values for A, B, and e_1, and $x = 0.5$ and $y \approx 2\sqrt{3} - 3 + 4\epsilon/3$ corresponding to our geodesics, Eq. (6) predicts $a_f < 1$ for $\epsilon \gtrsim \epsilon_{\rm crit} \sim 0.003$. However, for sufficiently large spins, the term $L_{\rm rad} - 2E_{\rm rad} \sim \epsilon^3 \log(\epsilon)$ is subdominant and $a_f > 1$. Numerical results confirm this expectation: in Table I, we show the BH spin a_f after absorbing the particle, taking into account radiation reaction. As can be seen, $a_f > 1$ already for $a = 0.9998$, corresponding to $\epsilon = 0.01 > \epsilon_{\rm crit}$. This is because, as already mentioned, Eqs. (9)–(11) overpredict the fluxes for $a \gtrsim 0.999$.

Moreover, even the result that $a_f < 1$ for $\epsilon \gtrsim \epsilon_{\rm crit}$ is questionable. Indeed, the fluxes down the horizon might destroy the BH before the particle is captured, while our code only calculates the fluxes at infinity. This is sufficient for our purposes because we used the code only to test the scaling (9)–(11), which is expected to hold *both* for the fluxes at infinity and down the horizon, since its derivation is generic. Once validated, that scaling implies that for sufficiently large spins both fluxes are smaller than the terms giving rise to naked singularities [i.e., the quadratic terms in Eq. (6)]. For $\epsilon \gtrsim \epsilon_{\rm crit}$, instead, the fluxes at infinity decrease the final spin to $a_{\rm fin} < 1$. However, in such a situation also the fluxes down the horizon, $L_{\rm rad,in}$ and $E_{\rm rad,in}$, are expected to be important (because the fluxes are produced when the particle sits at the light ring, which roughly corresponds to the maximum of the effective potential for gravitational waves), and could destroy the horizon before the particle is captured. In fact, the spin change is $\Delta a = L_{\rm rad,in} - 2E_{\rm rad,in} = E_{\rm rad,in}(1/\Omega_{\rm ph} - 2) \approx 2\sqrt{3}\epsilon E_{\rm rad,in}$, because $E_{\rm rad,in}/L_{\rm rad,in} = \Omega_{\rm ph} \approx 1/2 - (\sqrt{3}/2)\epsilon$. Since $\Omega_{\rm ph}$ is larger than the horizon's frequency $\Omega_{\rm hor} \approx 1/2 - \epsilon$, radiative emission is nonsuperradiant and $E_{\rm rad,in} > 0$; hence, $\Delta a > 0$. Thus, the ingoing fluxes increase a_f.

TABLE I. Initial and final BH spin after absorbing a particle with energy $E = \sqrt{2(1-a)}$ and angular momentum $L = b_{\rm ph} E(1\text{–}10^{-5})$, neglecting conservative SF effects, but not radiation reaction. We also show the final spin without radiation reaction ($a_f^{\rm JS}$) predicted by JS.

a	0.99	0.992	0.994	0.996	0.998	0.999	0.9998
a_f	0.882	0.928	0.961	0.984	0.997	0.9996	1.000 06
$a_f^{\rm JS}$	1.0043	1.0035	1.0026	1.0018	1.0009	1.000 45	1.000 09

So far we have shown that radiation reaction cannot prevent the formation of naked singularities, unless the impact parameter is extremely close to the light ring's impact parameter $b_{\rm ph}$. We will now show, however, that for all of JS' orbits the conservative SF is as important as the terms giving rise to naked singularities.

Let us consider a BH with gravitational radius $R_g = 2\,Gm/c^2$ in a curved background with curvature radius $\mathcal{L} \gg R_g$ [20]. The rigorous way of studying the motion of this BH is to set up a proper initial value formulation, but a reasonable alternative for practical purposes is to use a matched asymptotic expansion [21]. Near the BH (i.e., for $r < r_i$, r_i being a radius $\ll \mathcal{L}$), the metric is $g_{\rm internal} = g_{\rm BH} + H_1(r/\mathcal{L}) + H_2(r/\mathcal{L})^2 + \ldots$, where $g_{\rm BH}$ is the metric of an isolated BH and $H_1(r/\mathcal{L})$, $H_2(r/\mathcal{L})^2$ are corrections due to the "external" background. Far from the BH (i.e., for $r > r_e$, r_e being a radius $\gg R_g$), the metric is instead $g_{\rm external} = g_{\rm background} + h_1(R_g/\mathcal{L}) + h_2(R_g/\mathcal{L})^2 + \ldots$, i.e., the background metric plus perturbations due to the BH's presence. Because $R_g \ll \mathcal{L}$, there exists a region $r_e < r < r_i$ where both pictures are valid and the two metrics can be matched. Doing so, one finds that the BH equations of motion are [21]

$$u^\mu \nabla_\mu u^\nu = f^\nu_{\rm cons} + f^\nu_{\rm diss} + \mathcal{O}(R_g/\mathcal{L})^2, \qquad (14)$$

where ∇ is the connection of the background spacetime. The terms $f^\nu_{\rm cons}$ and $f^\nu_{\rm diss}$ are $\mathcal{O}(R_g/\mathcal{L})$, and are known as the conservative and dissipative SF. Remarkably, it turns out that Eq. (14) is the geodesic equation of a particle in a "perturbed" metric $\tilde{g} = g + h^R$, where h^R is a smooth tensor field of order $\mathcal{O}(R_g/\mathcal{L})$:

$$\tilde{u}^\mu \tilde{\nabla}_\mu \tilde{u}^\nu = 0 \qquad (15)$$

(the connection $\tilde{\nabla}$ and the four velocity $\tilde{u}^\mu$ being defined with respect to the perturbed metric $\tilde{g} = g + h^R$).

The dissipative SF amounts to the energy and angular-momentum fluxes considered earlier. Taking for instance the energy loss, Eq. (14) and $E = -p_t$ give $dE/d\tau = -m f_t^{\rm diss} = \mathcal{O}(R_g/\mathcal{L})^2$. Assuming now that the background spacetime is a BH with mass $M \sim \mathcal{L} \gg R_g$, and specializing to orbits near the horizon, one has $dt/d\tau \sim r_H/(r - r_H)$, which gives $dE/dt \sim (r - r_H)\mathcal{O}(R_g/\mathcal{L})^2$. A comparison of this scaling with our numerically validated scaling (12) shows that for a BH with $E \gg m$ the size entering the matched asymptotic expansion above (the "physical" size) is $R_g = 2\,GE/c^2$ and not $R_g = 2\,Gm/c^2$. This is no surprise, as the physical size associated with an ultrarelativistic BH is dictated by its energy and not by its mass, because in general relativity energy gravitates. Remarkably, however, we were able to test this fact with the numerical results presented earlier. Further evidence comes from boosting the Schwarzschild line element to the speed of light, keeping the total energy fixed. One gets the Aichelburg-Sexl metric, which depends on the total energy E and not on the rest mass [22]: this boosted BH absorbs particles within a distance $\sim E$ from it.

Because a BH's size is determined by $\max(E, m) \gtrsim \epsilon$, the conservative SF affects JS' analysis. This is easier to see from Eq. (15) [although the same result can be obtained from Eq. (14): see Ref. [23]]: because the metric "perturbation" h^R is $\mathcal{O}(R_g/\mathcal{L}) = \mathcal{O}(\epsilon)$, the effective potential for the radial motion differs from the "geodetic" one by $\mathcal{O}(R_g/\mathcal{L}) = \mathcal{O}(\epsilon)$ [23,24]. Therefore, $b_{\rm ph}$ changes by $\delta b_{\rm ph} = \mathcal{O}(R_g/\mathcal{L}) = \mathcal{O}(\epsilon)$. Because JS' orbits have $b_{\rm ph} - b = \mathcal{O}(R_g/\mathcal{L}) = \mathcal{O}(\epsilon)$, the conservative SF may prevent them from plunging into the horizon. This effect is intuitive: if the particle's size is $\sim\epsilon$, finite-size effects are important for impact parameters $b = b_{\rm ph} + \mathcal{O}(\epsilon)$.

A calculation of $\delta b_{\rm ph}$ is not doable with present technology [25], but we can estimate its sign. Because of frame dragging, for $a = 1 - 2\epsilon^2$ one has $b_{\rm ph} = 1/\Omega_{\rm ph} + \mathcal{O}(\epsilon)^2$. While the SF effect on $\Omega_{\rm ph}$ has not been calculated yet, the authors of Ref. [23] calculated the innermost stable circular orbit (ISCO) frequency shift for $a = 0$, and showed that the conservative SF increases $\Omega_{\rm ISCO}$. It therefore seems plausible that $\Omega_{\rm ph}$ should follow the same behavior. While approximate methods for calculating the conservative SF in Kerr spacetimes exist [26,27], they have problems for large spins, and the definitive answer to whether $b_{\rm ph}$ increases or decreases for $a \approx 1$ will only be available when a rigorous SF calculation [25] is performed. However, assuming that the $a = 0$ behavior of Ref. [23] holds also for $a \approx 1$, one obtains that $\Omega_{\rm ph}$ increases due to the SF, and therefore $b_{\rm ph}$ should decrease, possibly preventing the capture of the particles with dangerously large L and the formation of naked singularities.

In conclusion, we have shown that radiation reaction effects can prevent the formation of naked singularities only for some of the orbits for nonspinning particles around almost extremal Kerr BHs identified by JS. However, for all orbits capable of producing naked singularities, the conservative SF is non-negligible and seems to have the right sign to prevent the particles from being captured, thus saving the CCC.

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