Generalization of the Khinchin Theorem to Lévy Flights

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One of the most fundamental theorems in statistical mechanics is the Khinchin ergodic theorem, which links the ergodicity of a physical system with the irreversibility of the corresponding autocorrelation function. However, the Khinchin theorem cannot be successfully applied to processes with infinite second moment, in particular, to the relevant class of Lévy flights. Here, we solve this challenging problem. Namely, using the recently developed measure of dependence called Lévy correlation cascade, we derive a version of the Khinchin theorem for Lévy flights. This result allows us to verify the Boltzmann hypothesis for systems displaying Lévy-flight-type dynamics.

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Introduction.—Recently, ergodic properties of systems exhibiting anomalous behavior has attracted the growing attention of researchers in various fields of physics and related sciences. Ergodicity breaking was reported in blinking quantum dots [1,2]. A method of investigating weak ergodicity breaking for continuous-time random walks was introduced in [3]. This approach was further extended to describe ergodic properties of subdiffusion processes in the framework of the fractional Fokker-Planck equation [4,5]. Ergodicity of stochastic dynamics described by the generalized Langevin equations was studied in [6,7]; see also [8–11]. The relationship between ergodicity and irreversibility of anomalous systems was investigated in detail in [12]. For the analysis of the time average mean-square displacement of fractional Brownian motion see [13,14].

Verification of the Boltzmann ergodic hypothesis for a given system is one of the most fundamental problems in statistical mechanics. The key result in this field is the celebrated Khinchin theorem (KT) [15], which states that if the autocorrelation function vanishes at infinity (the property of irreversibility), then the process is ergodic. A good reconstruction of the role of Khinchin's approach to ergodic problem can be found in [16]. However, the KT can be successfully applied only under the assumption that the second moment (and thus the autocorrelation function) of the considered process is finite. This assumption is obviously fulfilled by many processes, in particular, by the relevant family of Gaussian processes.

Nonetheless, the situation gets more complicated while considering anomalous diffusion processes with infinite second moment, in particular, processes with α-stable marginal distributions [17,18]. These processes, being the natural generalization of Gaussian ones, are of great interest in physics, because of their fundamental role in the modeling of Lévy-flight-type dynamics. This kind of anomalous dynamics has been recently observed in a number of systems, including: animal foraging patterns, bulk mediated surface diffusion, transport in micelle systems or heterogeneous rocks, single molecule spectroscopy and wait-and-switch relaxation, to name only few (see [19] and references therein). Therefore, determining ergodic or even more chaotic properties of α-stable processes [17], which are the fundamental mathematical models of Lévy flights, is an important and timely problem.

In this Letter we develop a rigorous approach to study ergodicity of anomalous diffusion processes (Lévy flights). By the use of a new universal tool—the Lévy correlation cascade [20], which is an analogue of n-point correlation function for the processes with infinite second moment, we derive a version of the KT for Lévy flights. We illustrate the strength of the obtained theorem by applying it to a number of well-known Lévy flight processes (α-stable Ornstein-Uhlenbeck process, fractional α-stable noise, etc.), and present some numerical results. Finally, we indicate implications of the KT to three challenging problems of contemporary physics.

Lévy flights and correlation cascade.—A problem we are going to discuss now is how to verify ergodicity of a α-stable process $Y(t)$ of the general form

$$Y(t) = \int_{-\infty}^{\infty} K(t,x)dL_\alpha(x), \qquad t \in \mathbb{R}. \tag{1}$$

Here, $K(t,x)$ is the kernel function and $L_\alpha(x)$ is the symmetric α-stable Lévy motion with the Fourier transform $\langle e^{izL_\alpha(x)}\rangle = e^{-x|z|^\alpha}$, $0<\alpha<2$. We additionally assume that $Y(t)$ is stationary (for the explicit representation of the kernel $K(t,x)$ in the stationary case; see [21]). The assumption about stationarity of the process $Y(t)$ is absolutely crucial. Its physical meaning is obvious—the system is in thermal equilibrium. From the mathematical point of view, stationarity is necessary for the Birkhoff ergodic theorem [22] to apply. This in turn assures validity of the Boltzmann hypothesis [23].

It should be underlined that every stationary symmetric α-stable process (satisfying the additional very weak separability condition) can be represented in form (1) (see [18] for the details). Thus, the definition of $Y(t)$ is very general, and there is a wide range of physically relevant Lévy flight processes admitting representation (1). It includes the (fractional) α-stable Ornstein-Uhlenbeck processes, moving-average processes, and (fractional) α-stable noises.

Note that, in contrast to the Gaussian case ($\alpha = 2$), for α-stable processes with $0 < \alpha < 2$, the second moment is infinite. Thus, the autocorrelation function can no longer be used to determine ergodic properties of $Y(t)$. In particular, the KT cannot be applied. Therefore, it is necessary to develop a different mathematical tool, which will substitute the autocorrelation function in the α-stable case.

The Lévy correlation cascade corresponding to the process $Y(t)$ admitting representation (1), is defined as [20]

$$R(t_1, \ldots, t_n) = \int_{-\infty}^{\infty} \min\{|K(t_1, x)|, \ldots, |K(t_n, x)|\}^{\alpha} dx,$$

with $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{R}$. In particular, for the stationary process $Y(t)$, the function

$$R(0, t) = \int_{-\infty}^{\infty} \min\{|K(0, x)|, |K(t, x)|\}^{\alpha} dx \qquad (2)$$

is the α-stable analogue of the autocorrelation function. $R(0, t)$ will be called the Lévy autocorrelation function. As we will show, $R(0, t)$ plays a fundamental role in determining ergodic properties of Lévy flights. For more details on the general properties of Lévy correlation cascades, see [20,24,25].

Let us now derive the following main result of the Letter:

Khinchin theorem for Lévy flights.—If the Lévy autocorrelation function satisfies

$$\lim_{t\to\infty} R(0, t) = 0, \qquad (3)$$

then the stationary process $Y(t)$ is ergodic. Moreover, the Boltzmann hypothesis is true; i.e., the temporal and ensemble averages coincide

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T Y(t) dt = \langle Y(0) \rangle, \qquad (4)$$

provided that $\langle Y(0) \rangle$ is well defined.

To prove the above result, let us first note that the refined version of the classical Maruyama's mixing theorem [26,27] states that $Y(t)$ is ergodic if the Lévy measure ν_{0t} of the vector $(Y(0), Y(t))$ satisfies

$$\lim_{t\to\infty} \nu_{0t}\{(x, y): |xy| > \delta\} = 0 \quad \text{for every } \delta > 0. \qquad (5)$$

Next, by [28], the following relation between $R(0, t)$ and the Lévy measure ν_{0t} holds for every $\delta > 0$,

$$R(0, t) = c\delta^{\alpha} \nu_{0t}\{(x, y): \min\{|x|, |y|\} > \delta\}. \qquad (6)$$

Here, $c > 0$ is the appropriate constant. Next, let us choose $\epsilon > 0$ arbitrary small and denote by ν_0 the Lévy measure of $Y(0)$. Then, there exists $n \in \mathbb{N}$, such that $\nu_0\{x: |x| > n\} < \frac{\epsilon}{2}$. Moreover, we have

$$\begin{aligned}
&\nu_{0t}\{(x, y): |xy| > \delta\} \le \\
&\le \nu_{0t}\{(x, y): \min\{|x|, |y|\} > \delta/n\} + \nu_{0t}\{|x| > n \vee |y| > n\} \\
&\le \nu_{0t}\{(x, y): \min\{|x|, |y|\} > \delta/n\} + 2\nu_0\{x: |x| > n\} \\
&= \nu_{0t}\{(x, y): \min\{|x|, |y|\} > \delta/n\} + \epsilon.
\end{aligned}$$

Consequently, if $\lim_{t\to\infty} R(0, t) = 0$, then by (6) also condition (5) holds true, which in turn implies ergodicity of $Y(t)$. This shows that the sufficient condition for the Lévy flight process $Y(t)$ to be ergodic, is that the corresponding Lévy autocorrelation function $R(0, t)$ vanishes at infinity. Moreover, if $Y(t)$ is ergodic, then by the Birkhoff theorem [22] we obtain (4).

Since the classical KT was derived many years before Maruyama [26], Khinchin restricted his considerations only to the processes with finite second moment. Here, by the use of Lévy correlation cascade and Maruyama's theorem, we were able to extend the result of Khinchin to the processes with infinite second moment. The above theorem shows that $R(0, t)$ is a powerful mathematical tool for studying ergodic properties of α-stable processes, and that it is a proper analogue of the autocorrelation function. Its great advantage is the fact that $R(0, t)$ can be easily calculated for many relevant processes, as the following examples show:

Ornstein-Uhlenbeck process.—The first example is the α-stable Ornstein-Uhlenbeck process [17,18] defined as

$$Y_1(t) = \sigma \int_{-\infty}^{t} e^{-\lambda(t-x)} dL_{\alpha}(x), \qquad \lambda, \sigma > 0.$$

For $\alpha = 2$ we recover the classical Gaussian Ornstein-Uhlenbeck process. Using (2) with $K(t, x) = \sigma e^{-\lambda(t-x)} \times \mathbf{1}\{x < t\}$, we get that the Lévy autocorrelation function corresponding to $Y_1(t)$ satisfies $R(0, t) \propto e^{-\alpha\lambda t}$ as $t \to \infty$. Thus, using the KT for Lévy flights, we get that $Y_1(t)$ is ergodic. Equivalently, we obtain that the velocity process in the α-stable Klein-Kramers model [29] is ergodic.

In Fig. 1 (top panel) we observe a sample trajectory of $Y_1(t)$. The bottom panel of Fig. 1 depicts the behavior of its temporal average. Clearly $\lim_{T\to\infty} \frac{1}{T} \int_0^T Y_1(t) dt = 0 = \langle Y_1(0) \rangle$, which confirms validity of the Boltzmann hypothesis. In Fig. 2 we see the behavior of the Lévy autocorrelation function corresponding to $Y_1(t)$, calculated for various parameters α. In each case $R(0, t)$ decays to zero, which confirms ergodicity of $Y_1(t)$.

Moving-average processes.—The previously considered example belongs to the wider family of the so-called moving-average processes [17,18]. These processes admit the following representation

$$Y_2(t) = \int_{-\infty}^{t} f(t - x) dL_{\alpha}(x). \qquad (7)$$

Here, the kernel function f is assumed to be nonnegative, monotonically decreasing, and it satisfies the integrability condition $\int_{\mathbb{R}} f^{\alpha}(x) dx < \infty$, [18]. The Lévy autocorrelation function of $Y_2(t)$ yields $R(0, t) \propto \int_t^{\infty} f^{\alpha}(x) dx$, thus it

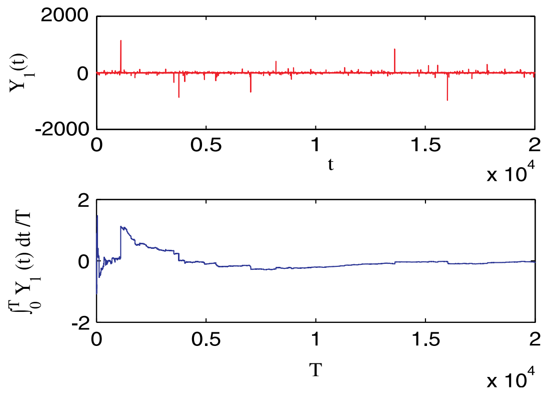

FIG. 1 (color online). In the top panel we observe a simulated sample trajectory of the α-stable Ornstein-Uhlenbeck process $Y_1(t)$. The bottom panel depicts the behavior of the temporal average corresponding to the simulated trajectory of $Y_1(t)$. Clearly, $\lim_{T\to\infty}\frac{1}{T}\int_0^T Y_1(t)dt = 0 = \langle Y_1(0)\rangle$, which confirms validity of the Boltzmann hypothesis. The parameters are: $\alpha = 1.2$, $\sigma = \lambda = 1$.

converges to zero as $t \to \infty$. Consequently, by the KT for Lévy flights, $Y_2(t)$ is ergodic.

α-stable noise.—The process of increments of the α-stable Lévy motion, defined as

$$l_\alpha(t) = L_\alpha(t+1) - L_\alpha(t), \qquad t \in \mathbb{N}, \tag{8}$$

is a stationary sequence of independent and identically distributed random variables. $l_\alpha(t)$ is called α-stable noise. It can be represented in the following way $l_\alpha(t) = L_\alpha(t+1) - L_\alpha(t) = \int_{-\infty}^{\infty} \mathbf{1}_{\{t<x<t+1\}} dL_\alpha(x)$. Therefore, by (2), the Lévy autocorrelation function of $l_\alpha(t)$ equals zero (this corresponds to the well-known property that independent random variables are uncorrelated). Thus, by the KT for Lévy flights, the α-stable noise is ergodic.

Fractional α-stable noise.—Let $0 < \alpha \leq 2$, $0 < H < 1$. Then the process

$$L_{\alpha,H}(t) = \int_{-\infty}^{\infty} [(t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}] dL_\alpha(x)$$

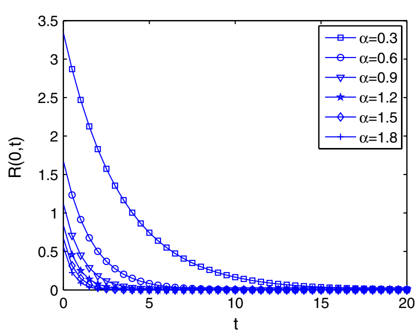

FIG. 2 (color online). Asymptotic behavior of $R(0,t)$ corresponding to the α-stable Ornstein-Uhlenbeck process $Y_1(t)$. For each α the function $R(0,t)$ tends to zero as $t \to \infty$, which confirms ergodicity of $Y_1(t)$.

is called the fractional α-stable motion [17,18]. Here $x_+ = \max\{x, 0\}$. $L_{\alpha,H}(t)$ is H self-similar with stationary increments. For $\alpha = 2$ it reduces to the well-known fractional Brownian motion, which was used by Mandelbrot and Van Ness to model long-range dependence phenomena [30]. Next, the process of increments

$$l_{\alpha,H}(t) = L_{\alpha,H}(t+1) - L_{\alpha,H}(t), \tag{9}$$

$t \in \mathbb{N}$, is called the fractional α-stable noise. Contrary to the previously considered noise $l_\alpha(t)$, the dependence between even very distant time points of $l_{\alpha,H}(t)$ is still very strong. Therefore, the fractional α-stable noise is often used to model phenomena displaying both Noah and Joseph effect [25,31,32]. After some tedious calculations, one proves that the Lévy autocorrelation function of $l_{\alpha,H}(t)$ yields $\lim_{t\to\infty} R(0,t) = 0$. Therefore, the KT for Lévy flights assures that the fractional α-stable noise is ergodic.

Harmonizable processes.—An important subclass of stable processes, for which we observe ergodicity breaking, is the class of harmonizable processes [17]. Every stable harmonizable process has the form $X_\alpha(t) = \mathrm{Re}\int_{\mathbb{R}} e^{itz} dW(z)$, where W is the appropriate α-stable stochastic measure. Contrary to the Gaussian case, stable harmonizable processes are never ergodic and cannot be represented in the moving-average form (7); see [17]. A complete description of nonergodic stable processes can be found in [33]. Applying the Maruyama's mixing theorem, we obtain from (5) and (6) that condition (3) does not hold for harmonizable processes. Ergodicity breaking for an exemplary harmonizable process is depicted in Fig. 3.

Concluding remarks.—In this Letter we have derived the KT for Lévy flights. This result allows us to determine ergodic properties and verify the Boltzmann hypothesis for the systems exhibiting anomalous Lévy flight behavior.

The main obstacle in studying ergodicity of Lévy flights is the divergence of the second moment. Therefore, the classical Khinchin approach based on the autocorrelation functions, can no longer be used. The method of determining ergodic properties of Lévy flights proposed in this Letter relies on the Lévy autocorrelation function, which

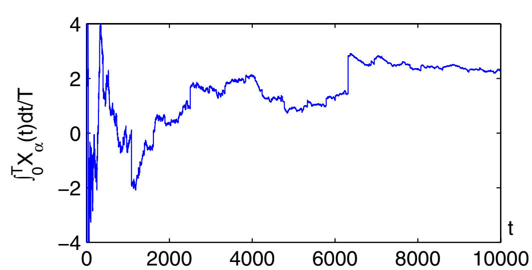

FIG. 3 (color online). The temporal average corresponding to the harmonizable process $X_\alpha(t) = \mathrm{Re}\int_{\mathbb{R}} e^{itz} dW(z)$. Here, W is the isotropic symmetric α-stable random measure with the control measure equal to $\gamma e^{-z} dz$, and γ is the uniform measure on the circle [18]. Since X_α is symmetric, we get that $\langle X_\alpha(t)\rangle = 0$. Thus, the time and ensemble averages do not coincide. Here $\alpha = 1.6$.

is the α-stable analogue of the standard autocorrelation function. We have shown that in the Lévy flight case, the irreversibility condition considered by Khinchin needs to be replaced by the similar one (3) given in terms of the Lévy autocorrelation function.

Verification of the Boltzmann hypothesis for a given physical system is one of the fundamental problems of statistical physics. Knowing that the average of a process parameter over time and the average over the space are the same, one can equivalently observe one realization for a long time or many independent short realizations. This is particularly important in the context of conducting physical experiments. Validity of the Boltzmann hypothesis allows us to optimize the experimental methods, since then it is enough to observe experimentally only one long trajectory of a process.

Our results show that the transport of light in special optical materials (Lévy glass) [34] is ergodic. This follows immediately from example (8) with $\alpha = 0.948 \pm 0.09$. Also, the motion of mRNA molecules inside bacterial cytoplasm [35], which can be described by the fractional stable motion [36], is ergodic. This is the consequence of example (9) with $\alpha = 1.85$. For the two above-mentioned experiments, as well as for the other experiments in which Lévy flight dynamics is observed, validity of the Boltzmann hypothesis is verified using the KT for Lévy flights.

Another interesting implication of the KT for Lévy flights concerns searching strategies. In their foraging patterns, animals tend to perform Lévy flights [37–39]. The advantage of such strategy is that it reduces over-sampling and thus optimizes the intermittent search. KT theorem for Lévy flight implies that Lévy searching strategies are ergodic. Since the successive jumps in such strategies are the independent α-stable random variables, example (8) assures that Lévy foraging patterns are ergodic. This shows that such animal strategies have another great advantage—since the dynamics is ergodic, every region of the scanned territory will be eventually visited by the animal [17]. This surprising result explains why Lévy strategies are advantageous and therefore so common in nature.

We underline that the introduced method of determining ergodic properties can be successfully applied also in the general case of stationary processes with infinitely divisible marginal distributions [17], including physically relevant subclasses (Pareto, gamma, Mittag-Leffler, and tempered stable).

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