

Full Counting Statistics of a Luttinger Liquid Conductor

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(Received 29 March 2010; published 13 December 2010)

Nonequilibrium bosonization technique is used to study current fluctuations of interacting electrons in a single-channel quantum wire representing a Luttinger liquid (LL) conductor. An exact expression for the time resolved full counting statistics of the transmitted charge is derived. It is given by the Fredholm determinant of the counting operator with a time-dependent scattering phase. The result has a form of counting statistics of noninteracting particles with fractional charges, induced by scattering off the boundaries between the LL wire and the noninteracting leads.

DOI: 10.1103/PhysRevLett.105.256802

PACS numbers: 73.23.-b, 73.40.Gk, 73.50.Td

Fluctuations are among the most fundamental concepts arising in statistical physics. In recent years, nonequilibrium noise was measured in a variety of electronic systems, such as quantum point contacts [1], diffusive mesoscopic conductors [2], and fractional quantum Hall edges [3]; see Ref. [4] for review. With the decrease of a sample size, characterization of current fluctuations by the second moment only becomes insufficient. This has triggered recent interest in higher-order correlation functions of current statistics. The third cumulant of the noise was measured in recent experiments [5]. A more complete characterization of current fluctuations is the full counting statistics (FCS), introduced by Levitov *et al.* [6]. This fascinating theoretical approach yields information about all moments of the number of electrons transferred (over a given time interval) through a terminal in a multiterminal system, current cross-correlations and entanglement, and large current fluctuations.

For noninteracting systems, the problem of FCS is well understood by now within several complementary approaches, including the Fredholm determinant formalism [6,7], the σ -model field-theoretical description [8], as well as the kinetic theory of fluctuations [9].

Much less remains known concerning fluctuations in interacting systems. This problem is of particular interest at low dimensions, where interaction affects the nature of the system in an essential way. This is the case for the FCS of current through a quantum dot in the Coulomb blockade regime [10], for the FCS of quantum impurities [11], and wires in an Ohmic environment [12], [13]. The interaction also affects dramatically the physics of one-dimensional (1D) fermionic systems where a strongly correlated state—Luttinger liquid (LL)—is formed [14,15]. Experimental realizations of LL include carbon nanotubes, semiconductor, metallic and polymer nanowires, as well as quantum Hall edges. Recent experiments studied the shot

noise in carbon nanotubes [16]. Nonequilibrium physics of carbon nanotubes and quantum Hall edges has been explored through tunneling spectroscopy [17] and Mach-Zehnder-interferometry [18] respectively. The LL description is also relevant to interacting 1D bosonic systems, cf., e.g., recent experiments on ultracold atomic gases that probe statistical properties of interference contrast, thus obtaining information on the full distribution of quantum noise [19].

Previous theoretical work on current fluctuations in LL mainly focused on the second moment (shot noise) [20]. The most intriguing observation was related to manifestations of fractional charges in shot noise. However, full understanding of the nature of charge transfer processes (in particular, of charge fractionalization [21]) requires the analysis of the FCS. The latter has been studied in a biased LL with an impurity [22]. While an analytical solution via the thermodynamic Bethe ansatz can be found [22], it is in general implicit and very cumbersome.

In this work we study the statistical properties of finite frequency current fluctuations in nonequilibrium LL conductor with arbitrary energy distribution of electrons. Employing the recently developed nonequilibrium bosonization technique [23], we find an exact solution of this problem. We show that the full distribution of current noise is closely related to the phenomenon of charge fractionalization. We demonstrate that the time resolved FCS of LL reduces to the one of a noninteracting model with fractional charges calculated below.

We consider a LL conductor geometry schematically shown in Fig. 1. Our goal is to solve the FCS problem in the presence of electron interaction. We assume that the spatially dependent interaction strength $g(x)$ vanishes outside the interval $[-L/2, L/2]$; this way of modeling leads was introduced in Refs. [24–26] to study the conductance of LL wire. To describe the FCS of charge transfer,

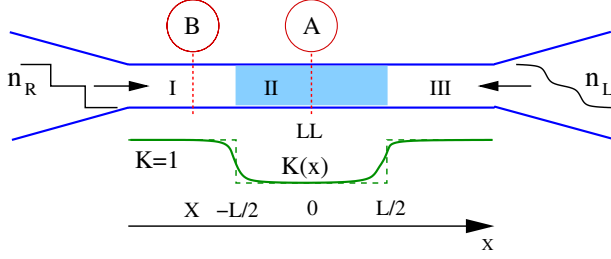


FIG. 1 (color online). Schematic view of noise detection in LL conductor. The positions of the charge detector inside (A) and outside (B) of interacting region are indicated. The x -dependent LL parameter $K(x)$ for smooth (solid line) and sharp (dashed line) boundaries is also shown.

we define a probability distribution function $p_\tau(n)$ for n electrons to pass through the cross section during the time interval τ (related to the noise moments at frequencies τ^{-1}) and a generating function $\kappa(\lambda) = \sum_{n=-\infty}^{\infty} p_\tau(n) e^{in\lambda}$.

Let us first discuss the noninteracting case, where $\kappa(\lambda)$ has been calculated by means of Landauer approach [6]. For an ideal quantum wire (with no scattering inside the wire) the generating function of the FCS is given by

$$\kappa(\lambda) = \Delta_R[\delta_R(t)] \Delta_L[\delta_L(t)]. \quad (1)$$

Here, $\eta = R, L$ labels right and left movers and

$$\Delta_\eta[\delta_\eta(t)] = \det[1 + (e^{-i\delta_\eta} - 1)n_\eta] \quad (2)$$

is a Fredholm determinant (of the Toeplitz type) of the counting operator built of fermionic distribution function $n_\eta(\epsilon)$ and a time-dependent scattering phase $\delta_\eta(t)$, with ϵ and t understood as canonically conjugate variables. In Eq. (1) and below we assume normalization of the determinant Δ_η to its value for equilibrium, zero-temperature distribution. The phase $\delta_\eta(t)$ is given by

$$\delta_\eta(t) = \lambda \eta w_\tau(t, 0), \quad (3)$$

where we have defined a window function $w_\tau(t, \tilde{t}) = \theta(\tilde{t} - t) - \theta(\tilde{t} - t - \tau)$. We use the convention that in formulas η is understood as $\eta = \pm 1$ for right or left movers. In the long time limit the Fredholm determinant (2) can be easily evaluated analytically; in a more general situation it can be efficiently studied numerically [27]. Below we show that for the interacting case, the generating function of the FCS obeys the form of Eq. (1). All electron interaction effects are encoded in the time-dependent scattering phases $\delta_\eta(t)$.

On the microscopic level the problem is described by the Keldysh action $S = S_0[\psi] + S_{ee}[\psi]$, written in terms of fermionic fields [28],

$$S_0[\psi] = i \sum_\eta \int_c dt \int dx \psi_\eta^\dagger \partial_\eta \psi_\eta,$$

$$S_{ee}[\psi] = - \sum_\eta \int_c dt \int dx g(x) (\rho_\eta \rho_{-\eta} + \rho_\eta \rho_\eta).$$

Here $\rho_\eta = \psi_\eta^\dagger \psi_\eta$ are density fields, $\partial_\eta = \partial_t + \eta v \partial_x$, v is the Fermi velocity. The nonequilibrium bosonization

approach allows us to reformulate this theory in terms of bosonic (density) fields. The interacting part of the action, S_{ee} , is already expressed in terms of density modes ρ_η . Following rotation in Keldysh space $S_{ee} = \sum_{\eta\eta'} \int dt dx g(x) \rho_\eta \bar{\rho}_{\eta'}$, while the free part of the action, where information concerning the state of the noninteracting fermionic system is encoded, reads

$$S_0 = \sum_\eta (-\rho_\eta \Pi_\eta^{-1} \bar{\rho}_\eta - i \ln Z_\eta[\bar{\chi}_\eta]). \quad (4)$$

Here we have decomposed the bosonic variables into classical and quantum components, $\rho, \bar{\rho} = (\rho_+ \pm \rho_-)/\sqrt{2}$, where the indices $+$ and $-$ refer to the two branches of the Keldysh contour; Π_η^a is the advanced component of polarization operator, and $Z_\eta[\bar{\chi}_\eta]$ is a partition function of free fermions moving in the field

$$\bar{\chi}_\eta = \Pi_\eta^{-1} \bar{\rho}_\eta,$$

$$\Pi_\eta^a(\omega, q) = \eta q / 2\pi(\eta v q - \omega - i0). \quad (5)$$

Expansion of $Z_\eta[\bar{\chi}_\eta]$ in $\bar{\chi}_\eta(t, x)$ generates an infinite series, $i \ln Z_\eta[\bar{\chi}_\eta] = \sum_n (-1)^{n+1} \bar{\chi}_\eta^n \mathcal{S}_{\eta n} / n$, governed by irreducible fermionic density correlation functions, $\mathcal{S}_{\eta n}(t_1, x_1; \dots; t_n, x_n) \equiv \langle\langle \rho_{\eta 1} \rho_2 \dots \rho_{\eta n} \rangle\rangle$, representing cumulants of quantum noise [6].

To find the generating function for the FCS of charge transferred through a cross-section $x = X$ during a time interval τ , one needs to calculate the correlation function

$$\kappa(\lambda, X) = \langle e^{i\lambda Q(\tau, X)} e^{-i\lambda Q(0, X)} \rangle, \quad (6)$$

where $Q(t, X) = \int_{-\infty}^X (\rho_R(x) + \rho_L(x)) dx$ is the electric charge situated on the left side of the point X at the time t . Note a similarity between the problem of FCS and that of tunneling spectroscopy [23]. The latter amounts to evaluation of a single-particle Green function. In both cases one needs to find a correlation function of exponentials of bosonic operators that are linear combinations of the right and left density fields. A major difference stems from the fact that the operators $e^{\pm i\lambda Q}$ in Eq. (6) contain a sum of left and right densities in the exponential, whereas the fermionic operators $\psi_\eta, \psi_\eta^\dagger$ contain only ρ_η .

We now proceed with calculation of the generating function (6); technical details are outlined in the supplementary material [29]. We find that the FCS generating function has the form of Eq. (1), as in the case of free fermions. The time-dependent scattering phases $\delta_\eta(t)$ are expressed through plasmon reflection and transmission coefficients r_η, t_η at the left (I/II; $\eta = L$) and right (II/III; $\eta = R$) boundaries; $r_\eta^2 + t_\eta^2 = 1$. Multiple scattering off these boundaries gives rise to an infinite sequence of scattering phase pulses [cf. Equation (7)]. If the boundary is smooth (on the scale of the plasmon wave length), there is no plasmon reflection, $r_\eta \approx 0$. In the opposite limit of a sharp boundary we have $r_\eta = (1 - K)/(1 + K)$.

In contrast with the free fermion case, the statistics of current fluctuations in the LL conductor depend on the

position of the measuring device (see Fig. 1). Let us first consider the case where the current fluctuations are measured in the middle of an interacting region ($X = 0$, position A in Fig. 1). We thus find [29] that time-dependent scattering phases consist of a sequence of pulses

$$\delta_\eta(t) = \sum_{n=0}^{\infty} \delta_{\eta,n} w_\tau(t, t_n) \quad (7)$$

with partial phase shifts

$$\delta_{\eta,2n} = \eta \lambda t_{-\eta} \sqrt{K} r_\eta^n r_{-\eta}^n \equiv \eta \lambda e_{\eta,2n}^* \quad (8)$$

$$\delta_{\eta,2n+1} = \eta \lambda t_{-\eta} \sqrt{K} r_\eta^{n+1} r_{-\eta}^n \equiv \eta \lambda e_{\eta,2n+1}^* \quad (9)$$

The beginning of the n th pulse occurs at time $t_n = (n + 1/2 - 1/2K)L/u$. The sequence of pulses $\delta_\eta(t)$ is shown in Fig. 2 for the cases of sharp and smooth boundaries between the interacting and noninteracting regions.

In the limit of very low frequencies ($\tau \gg L/u$) all pulses (7) overlap and the scattering phase can be approximated by $\delta_\eta(t) \simeq w_\tau(t, 0) \sum_{n=0}^{\infty} \delta_{\eta,n}$. In this limit of long wavelength the interacting or noninteracting boundaries appear to be sharp. This yields $\sum_{n=0}^{\infty} \delta_{\eta,n} = \eta \lambda$. As expected [24–26] the effects of interaction in this limit disappear and one is back to the FCS of free fermions [6].

In the opposite case ($\tau \ll L/u$) the interference between different plasmon pulses may be neglected and the result splits into a product

$$\Delta_\eta[\delta_\eta(t)] \simeq \prod_{n=0}^{\infty} \Delta_{\eta\tau}(\delta_{\eta,n}), \quad \delta_{\eta,n} \equiv \eta \lambda e_{\eta,n}^* \quad (10)$$

with each factor representing a contribution of a single phase pulse $\delta_{\eta,n}(t) = \delta_{\eta,n} w_\tau(t, 0)$. Remarkably, plasmon scattering gives rise to charge fractionalization, which splits the scattering phase into an infinite series of pulses.

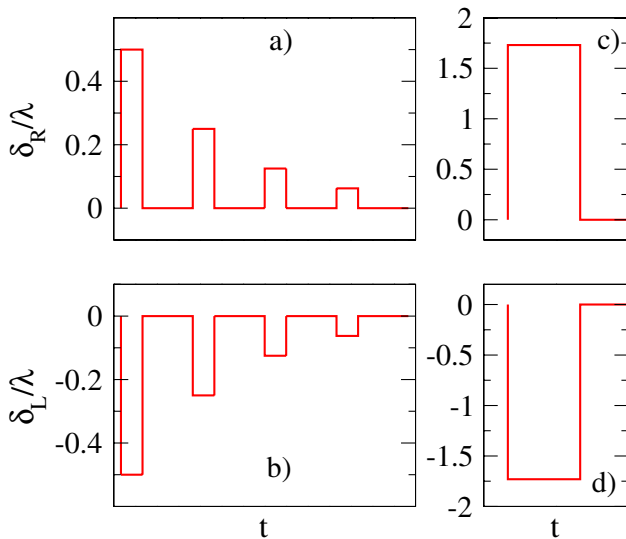


FIG. 2 (color online). Phases δ_η entering Eq. (1) for the measuring device located at $X = 0$, for sharp (a),(b) and adiabatic (c),(d) boundaries ($K = 1/3$).

On a technical level, the Fredholm determinant of the counting operator is now replaced by an infinite product of determinants, each calculated with a corresponding scattering phase. The FCS of the LL then becomes a superposition of FCS of noninteracting electrons with fractional charges $e_{\eta,n}^*$. For the case of smooth boundaries we get only one fractional charge, $e_{\eta,0}^* = \sqrt{K}$. In the opposite limit of sharp boundaries, we obtain the sequence of fractional charges of the form $e_{\eta,n}^* = 2K(1-K)^n/(1+K)^{n+1}$.

Focusing on the regime where the counting interval τ is long compared to the inverse energy scale of the distributions $n_\eta(\epsilon)$, we express $\Delta_{\eta\tau}(\delta)$ in the form

$$\ln \Delta_{\eta\tau}(\delta) = \tau \int \frac{d\epsilon}{2\pi} \{ \ln[1 + (e^{-i\delta} - 1)n_\eta(\epsilon)] + i\delta\theta(-\epsilon) \}. \quad (11)$$

Substituting (11) in Eqs. (1), (10) and expanding in δ , one can find explicitly all moments of noise.

Consider now the FCS outside the interacting region, e.g., at $X < -L/2$ (B in Fig. 1). The analysis follows closely the one outlined above, hence we present only the final result. The right and left scattering phases are

$$\begin{aligned} \delta_R(t) &= \lambda \left[w_\tau\left(t, \frac{X}{v}\right) - r_L w_\tau\left(t, \frac{L+X}{v}\right) \right. \\ &\quad \left. + r_R t_L^2 \sum_{n=0}^{\infty} r_L^n r_R^n w_\tau\left(t, \frac{L+X}{v} + \frac{2(n+1)L}{u}\right) \right], \\ \delta_L(t) &= -\lambda t_L t_R \sum_{n=0}^{\infty} r_L^n r_R^n w_\tau\left(t, \frac{2nL}{u} - \frac{X}{v} + \frac{v-u}{vu}L\right). \end{aligned}$$

Current fluctuations measured in the noninteracting region at finite frequency ($\tau \ll L/u$) differ from those at the interacting part of LL (8). Here too the FCS takes the form of superposition of noninteracting FCS's, but with a new set of fractional charges:

$$\begin{aligned} e_{R,0}^* &= 1, & e_{R,1}^* &= -r_L, & e_{R,n+2}^* &= t_L^2 r_L^n r_R^{n+1}, \\ e_{L,n}^* &= t_L t_R r_L^n r_R^n & (n &= 0, 1, \dots). \end{aligned} \quad (12)$$

In the universal ultra-low-frequency limit ($\tau \gg L/u$) the effects of interaction again disappear, and the result for the FCS of free fermions is recovered.

To conclude, we provide a brief overview of the physics discussed here. We have calculated the FCS in a LL conductor. The result is cast in terms of a Fredholm determinant of the counting operator. The boundaries between the interacting and noninteracting regions of the wire give rise to (in general, multiple) plasmon scattering. This generates plasmon wavelets corresponding to fractionally charged “particles.” As a result, for a counting time interval τ small compared with the plasmon time-of-flight through the interacting region, L/u , the FCS is a superposition of counting statistics of noninteracting particles with fractional charges $e_{\eta,n}^*$. Let us stress that the FCS contains more complete information about the system than the second moment (noise). As an example, for a symmetric

system ($r_L = r_R$) the noise acquires the factor K due to interaction. By itself, this is not sufficient to make a conclusion about the value of fractional charges and the character of correlations between them.

The fractionalization process that manifests itself in the FCS calculated above is solely due to the interacting or noninteracting boundary scattering [25]. Fractionalization due to tunneling into a LL [30] does not show up here: technically this is since the operator Q in Eq. (6) is determined by a sum of ρ_L and ρ_R .

How is the notion of charge fractionalization compatible with charge quantization? The above analysis was based on the bosonization approach with the density ρ_η slowly varying on the scale of the inverse Fermi wave vector k_F^{-1} . This corresponds to the situation where the measurement procedure is smooth on the scale k_F^{-1} . In this situation the above results are valid for any counting field λ : our detector may count a fractional charge residing in some volume of space. In the opposite limit, where the spatial resolution of our measurement is sharp on the scale k_F^{-1} , the measured charge should be integer, i.e., the FCS should satisfy $\kappa(\lambda) = \kappa(\lambda + 2\pi)$. In the bosonization formalism, charge quantization emerges by taking into account fast oscillatory contributions to the density ρ_η [31]. This will not affect the moments of FCS (which are obtained as derivatives of $\kappa(\lambda)$ at $\lambda = 0$) and, more generally, the form of $\kappa(\lambda)$ in the range $[-\pi, \pi]$. Beyond this interval, the FCS will be continued periodically, in agreement with the charge quantization requirement.

This work was supported by GIF, DFG Center for Functional Nanostructures, Einstein Minerva Center, US-Israel BSF, ISF, Minerva Foundation, DFG SPP 1285, EU GEOMDISS, and Rosnauka 02.740.11.5072.

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