Explosive Percolation Transition is Actually Continuous

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Recently a discontinuous percolation transition was reported in a new "explosive percolation" problem for irreversible systems [D. Achlioptas, R. M. D'Souza, and J. Spencer, Science **323**, 1453 (2009)] in striking contrast to ordinary percolation. We consider a representative model which shows that the explosive percolation transition is actually a continuous, second order phase transition though with a uniquely small critical exponent of the percolation cluster size. We describe the unusual scaling properties of this transition and find its critical exponents and dimensions.

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Introduction.—Percolation is one of the basic notions in statistical and condensed matter physics [1]. When one increases progressively the number of connections between nodes in a network, above some critical number (percolation threshold) a giant connected (percolation) cluster emerges in addition to finite clusters. This percolation cluster contains a finite fraction of nodes and links in a network. The percolation transition was widely believed to be a typical continuous phase transition for various networks architectures and space dimensionalities [2–4], so it shows standard scaling features, including a power-law size distribution of finite cluster sizes at the percolation threshold. Recently, however, it was reported that a remarkable percolation problem exists in which the percolation cluster emerges discontinuously and already contains a finite fraction of nodes at the percolation threshold [5]. This conclusion was based on simulations of a model in which each new connection is made in the following way: choose at random two links that could be added to the network, but add only one of them, namely, the link connecting two clusters with the smallest product of their sizes. To emphasize this surprising discontinuity, this kind of percolation was named "explosive" [5]. Further investigations of "explosive percolation" in this and similar systems, also mainly based on numerical simulations, supported this strong result but, in addition, surprisingly for discontinuous phase transitions, revealed scaling behavior [6–13], in particular, power-law critical distributions of cluster sizes [9,10,12] resembling those found in continuous percolation transitions. This self-contradicting combination of discontinuity and scaling have made explosive percolation one of the challenging and urgent issues in the physics of disordered systems.

Here we resolve this confusion. We show that there is not actually any discontinuity at the explosive percolation threshold, contrary to the conclusions of the previous investigators. We consider a representative model demonstrating this new kind of percolation and show that the explosive percolation transition is a continuous, second-order phase transition but, importantly, with a uniquely small critical exponent $\beta \approx 0.0555$ of the percolation cluster size.

Model.—One of the simplest systems in which classical percolation takes place is as follows. Start with N unconnected nodes, where N is large, and at each step add a connection between two uniformly randomly chosen nodes. In essence, this is a simple aggregation process [14], in which at each step, a pair of clusters, to which these nodes belong, merge together [Fig. 1(a)]. If we introduce "time" t as the ratio of the total number of added links in this system, L, and its size N, i.e., $t = L/N$, then

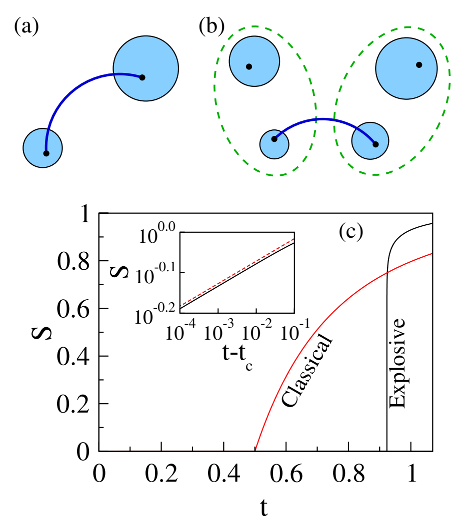

FIG. 1 (color online). (a) In classical percolation, at each step, two randomly chosen nodes are connected by a new link. If these nodes belong to different clusters, these clusters merge. (b) In the explosive percolation model, at each step, two pairs of nodes are chosen at random, and for each of the pairs, the node belonging to the minimal cluster is selected. These two nodes (and so their clusters) are connected by a new link. (c) The relative size S of a percolation cluster versus t for explosive percolation obtained by simulation of the model with 2×10^9 nodes (1000 runs) and, for comparison, $S(t)$ for classical percolation. Inset: the log-log plot S vs t for this data.

the percolation cluster of relative size S emerges at the percolation threshold $t_c = 1/2$ and grows with t in the following way: $S \sim \delta^{\beta}$, where $\delta = |t - t_c|$ and $\beta = 1$. At $t = t_c$, the cluster size distribution $n(s)$ (fraction of finite connected components of s nodes) in this classical problem is power law $n(s) \sim s^{-\tau}$ with $\tau = 5/2$ [1].

In this Letter, we consider a direct generalization of this process. Namely, at each step, we sample twice: (i) choose two nodes uniformly at random and compare the clusters to which these nodes belong; select that node of the two ones, which belongs to the smallest cluster; (ii) choose a second pair of nodes and, as in (i), select the node belonging to the smallest of the two clusters; and (iii) add a link between the two selected nodes thus merging the two smallest clusters [Fig. 1(b)]. Repeat this procedure again and again. Note that in (i) and (ii), a cluster can be selected several times. This is the case for the percolation cluster. These rules contain the key element of other explosive percolation models, e.g., model [5]. Namely, for merging, select the minimal clusters from a few possibilities. Importantly, our procedure provides even more stringent selection of small components for merging than model [5] since it guarantees that the product of the sizes of two merging clusters is the smallest of the four possibilities [each of the first pair of chosen nodes (i) may connect with any node of the second pair (ii)] in contrast to selection from only two possibilities in model [5]. Consequently, if we show that the transition in our model is continuous, than model [5] also must have a continuous transition. One should stress that the explosive percolation processes are irreversible in stark contrast to ordinary percolation. In the latter, one can reach any state either adding or removing connections. For explosive percolation, only adding links makes sense, and an inverse process is impossible.

We simulated this irreversible aggregation process for a large system of 2×10^9 nodes. When plotted over the full time range, the obtained dependence $S(t)$ shows what seems to be discontinuity at the critical point t_c [Fig. 1(c)] similar to previous results. On the other hand, the inspection of the log-log plot S versus $t - t_c$ [Fig. 1(c), inset] reveals that, surprisingly, the obtained dependence $S(t)$ can be described by the power law $S \propto (t - t_c)^{\beta}$ which indicates a continuous transition, in contrast to the previous investigations. This data still do not allow us to completely rule out a discontinuity, since we actually only succeed to check that the law $S_0 + b(t - t_c)^{\beta}$ with $S_0 < 0.05$ fits our data. This shows that for a definite conclusion, even so large a system is not sufficient, and a discontinuity can be ruled out or validated only by analytical arguments for the infinite size limit.

Equations.—We address this problem analytically and numerically by considering the evolution of the size distribution $P(s)$ for a finite cluster of s nodes to which a randomly chosen node belongs, $P(s) = sn(s)/\langle s \rangle$, where $\langle s \rangle$ is the average cluster size (the ratio of the number of nodes and the total number of clusters). This distribution satisfies the sum rule $\sum_s P(s) = 1 - S$. Another important characteristic in this model is the probability $Q(s)$ that if we choose at random two nodes, then the smallest of the two clusters to which these nodes belong is of size s. $Q(s)$ provides us with the size distribution of merging clusters. Here $\sum_s Q(s) = 1 - S^2$. If we introduce the cumulative distributions $P_{\text{cum}}(s) \equiv \sum_{u=s}^{\infty} P(u)$ and $Q_{\text{cum}} \equiv \sum_{u=s}^{\infty} Q(u)$, then probability theory gives $Q_{\text{cum}}(s) + S^2 = [P_{\text{cum}}(s) + S]^2$. Consequently

$$\begin{aligned} Q(s) &= [P_{\text{cum}}(s) + P_{\text{cum}}(s+1) + 2S]P(s) \\ &= [2 - 2P(1) - 2P(2) - \ldots - 2P(s-1) - P(s)]P(s), \end{aligned} \tag{1}$$

that is $Q(s)$ is determined by $P(s')$ with $s' \leq s$. The evolution of these distributions in the infinite system is exactly described by the master equation

$$\partial P(s,t)/\partial t = s \sum_{u+v=s} Q(u,t)Q(v,t) - 2sQ(s,t), \tag{2}$$

which generalizes the standard Smoluchowski equation. The only difference from the classical percolation problem [14] is $Q(s,t)$ instead of $P(s,t)$ on the right-hand side of this equation. Thus we have a chain of coupled equations, which should be solved analytically or numerically.

Numerical solution.—To find a numerical solution, first solve the first equation of the chain, which gives $P(1,t)$. Substitute this result into the second equation and solve it, which gives $P(2,t)$, and so on. In this way we find numerically the distributions $P(s,t)$ and $Q(s,t)$ at any t for infinite N. Solving 10^6 equations gives the evolution of these distributions for $1 \leq s \leq 10^6$ and $S(t) \cong 1 - \sum_{s=1}^{10^6} P(s,t)$. The log-log plot [Fig. 2] shows that the obtained $S(t)$ dependence is well described by the power law $S \propto \delta^{\beta}$ with $\delta = t - t_c$, $t_c = 0.923\,207\,508(2)$, and small $\beta = 0.0555(1)$ (which is very close if not equal to $1/18 = 0.055\,55\ldots$). Here we find t_c as the point at which $P(s)$ is power law over the full range of s; see below. To check the correctness and precision of our calculations,

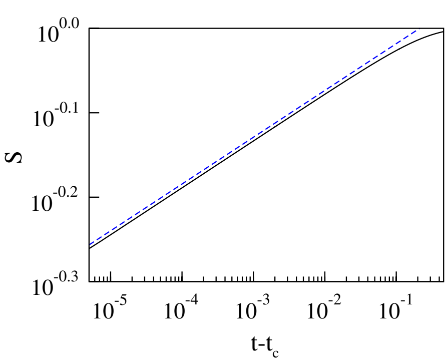

FIG. 2 (color online). Log-log plot S vs $t - t_c$ obtained by solving Eq. (2).

we repeated them for ordinary percolation and obtained the classical results with the same precision as for our model. Although the small exponent β makes it difficult to approach the narrow region of small S, fitting this data by the law $S_0 + b\delta^{\beta}$, we find that S_0 is smaller than 0.005. This supports our hypothesis that the transition is continuous, but still does not prove it. Moreover, both our extensive simulations and the numerical solution results clearly demonstrate that the analysis of the $S(t)$ data cannot validate or rule out a discontinuity.

Figure 3(a) shows the evolution of the distribution $P(s, t)$, which we compare with the evolution of this distribution for ordinary percolation, Fig. 3(b). The difference is strong at $t < t_c$, where the distribution for explosive percolation has a bump, but above t_c the behaviors are similar. The distribution function $Q(s, t)$ evolves similarly to $P(s, t)$ in the full time range. At the critical point, we find power law $P(s) \sim s^{1-\tau}$ and $Q(s) \sim s^{3-2\tau}$ in the full range of s (6 orders of magnitude), where $\tau = 2.04762(2)$, which is close to 2, as in Refs. [9,10,12], in contrast to $\tau = 5/2$ for classical percolation. The first moments of these distributions, $\langle s\rangle_P \equiv \sum_s sP(s)$ (the mean size of a finite cluster to which a randomly chosen node belong) and $\langle s\rangle_Q \equiv \sum_s sQ(s)$, demonstrate power-law critical singularities $\langle s\rangle_P \sim |\delta|^{-\gamma_P}$ and $\langle s\rangle_Q \sim |\delta|^{-\gamma_Q}$, where exponents $\gamma_P = 1.0111(1)$ and $\gamma_Q = 1.0556(5)$ both below and above the transition. Note that $\gamma_P > 1$ in contrast to ordinary percolation, where the mean-field value of exponent γ is 1. Figure 3(c) shows the set of time dependencies of $P(s, t)$ for fixed cluster sizes [the time dependencies of $Q(s, t)$ are similar]. These dependencies strongly differ from those for ordinary percolation [Fig. 3(d)] in the following respect. The peaks in Fig. 3(c) for explosive percolation are below t_c, while the peaks in Fig. 3(d) for ordinary percolation are symmetrical with respect to the critical point at large s.

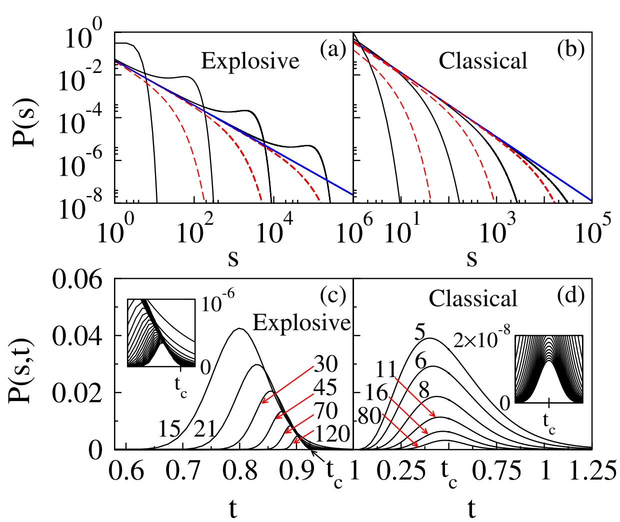

FIG. 3 (color online). Solutions of Eq. (2) for the infinite system. (a) The evolution of the distribution $P(s)$ below and at (solid lines) and above (dashed lines) the percolation threshold for explosive percolation. (b) The evolution of $P(s)$ for ordinary (classical) percolation. (c) Dependence of $P(s, t)$ on t for a set of cluster sizes s for explosive percolation. Numbers on curves indicate s. (d) $P(s, t)$ versus for normal percolation. The insets show the $P(s, t)$ curves for large values of s.

The inspection of these numerical results in the critical region reveals a scaling behavior typical for continuous phase transitions, $P(s, t) = s^{1-\tau} f(s\delta^{1/\sigma})$ and $Q(s, t) = s^{3-2\tau} g(s\delta^{1/\sigma})$, respectively, where $f(x)$ and $g(x)$ are scaling functions, and $\sigma = (\tau - 2)/\beta$, which is a standard scaling relation. One should stress that these functions are quite unusual. Figure 4 shows the resulting scaling functions and, for comparison, the scaling function for ordinary percolation. Remarkably, $f(x)$ and, especially, $g(x)$, obtained at $t < t_c$, are well fitted by Gaussian functions. These functions differ dramatically from the monotonously decaying exact scaling function $e^{-2x}/\sqrt{2\pi}$ for ordinary percolation. Effective elimination of the smallest clusters in this merging process results in the minima of the scaling functions at $x = 0$. On the other hand, the stunted emergence of large clusters results in the particularly rapid decay of these functions at $x \gg 1$.

Analytical treatment.—The key point of our study is the following strict analytical derivation. We start from the fact that in this problem the cluster size distributions are power law at the percolation threshold. We observed these power laws over 6 orders of magnitude, and they were observed in works [9,10,12] though in less wide range of s [15]. Now we strictly show that if the cluster size distribution is power law at the critical point, $P(s, t_c) \propto s^{1-\tau}$, then this transition is continuous. Introducing generating functions, $\rho(z) \equiv \sum_{s=1}^{\infty} P(s)z^s$ and $\sigma(z) \equiv \sum_{s=1}^{\infty} Q(s)z^s$, where $\rho(z = 1) = 1 - S$ and $\sigma(z = 1) = 1 - S^2$, we represent Eq. (2) in the form

$$\partial[1 - \rho(z, t)]/\partial t = -\partial[1 - \sigma(z, t)]^2/\partial \ln z. \quad (3)$$

Above the percolation threshold, at large s, we have $Q(s) \cong 2SP(s)$; see Eq. (2). Consequently, for z close to 1, we get $1 - \sigma(z) \cong 2S[1 - \rho(z) - S/2]$, and so in this region, Eq. (3) leads to the equation

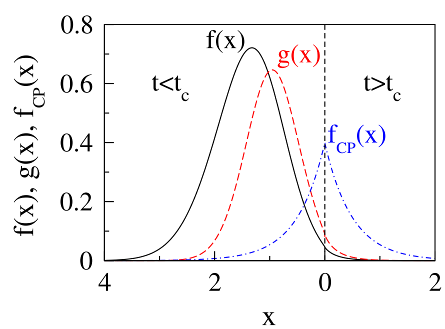

FIG. 4 (color online). Scaling functions $f(x)$ and $g(x)$ for explosive percolation for $t < t_c$ (left) and $t > t_c$ (right), and, for comparison, the scaling function $f_{CP}(x)$ for ordinary percolation.

TABLE I. Percolation thresholds, critical exponents, and fractal and upper critical dimensions.

	t_c	β	τ	σ	γ_P	γ_Q	d_f	d_u
Ordinary Percolation	1/2	1	5/2	1/2	1	-	4	6
Explosive Percolation	0.923 207 508(2)	0.0555(1)	2.047 62(2)	0.857(3)	1.111(1)	1.0556(5)	2.333(1)	2.445(1)

$$\partial\rho(z,t)/\partial t = 8S^2(t)[\rho(z,t) - 1 + S(t)/2]\partial\rho(z,t)/\partial \ln z. \tag{4}$$

We use the asymptotics $P(s,t_c) \cong f(0)s^{1-\tau}$ as the initial condition for this equation. For the generating function, this gives the singularity $1-\rho(z,t_c) \cong$ analytic terms $- f(0)\Gamma(2-\tau)(1-z)^{\tau-2}$ at $z=1$. We assume that $S \cong B\delta^\beta$, i.e., that the transition is continuous, substitute this expression into Eq. (4), and solve this equation using our initial condition. In implicit form, the solution is

$$\ln z = \frac{8B^2}{1+2\beta}\left[1-\rho-\frac{B}{2}\frac{1+2\beta}{1+3\beta}\delta^\beta\right]\delta^{1+2\beta} - \left[\frac{f(0)|\Gamma(2-\tau)|}{1-\rho}\right]^{-1/(\tau-2)}. \tag{5}$$

At $z=1$, this readily leads to the relation $\tau = 2 + \beta/(1+3\beta)$ [and so $\sigma = 1/(1+3\beta)$] which validates our assumption that $S \propto \delta^\beta$. This scaling relation agrees with our numerical results. Relation (5) allows us express B in terms of $f(0)$ and τ. Substituting our numerical results, $f(0) = 0.046\,18(2)$ and $\tau = 2.047\,62(2)$, into this expression gives $B = 1.075$, which agrees with the corresponding value obtained by solving Eq. (2) numerically. So the results of this report are self-consistent. Our results are summarized in Table I. Assuming a scaling form for the distributions gives $\gamma_P = 1 + 2\beta$ and $\gamma_Q = 1 + \beta$, which agree with our numerical solution of Eq. (2). Furthermore, applying standard scaling relations [1], we calculate the fractal dimension for this model, $d_f = 2/\sigma = 2(1+3\beta)$, and the upper critical dimension, $d_u = d_f + 2\beta = 2(1+4\beta)$. The latter determines the finite size effect: $t_c(\infty) - t_c(N) \propto N^{-2/d_u}$, where $2/d_u = 0.818(1)$. Interestingly, the obtained fractal and upper critical dimensions are less than 3. They are much smaller than those for ordinary percolation, which are 4 and 6, respectively. Our model is infinite dimensional, i.e., mean-field theories must work, which makes the observed smallness of exponent β particularly remarkable. Furthermore, our model allows a natural generalization. Let each minimal cluster for merging be selected of m possibilities, $m \geq 1$. We found that with increasing m, t_c approaches 1 and β rapidly decreases with m, but the transition remains continuous.

Conclusions.—We have shown that the explosive percolation transition is actually continuous. It is the smallness of the β exponent for the size of the percolation cluster that makes it virtually impossible to distinguish this phase transition from a discontinuous one even in very large systems. Indeed, suppose that $N = 10^{18}$ and $\beta \approx 1/18$. The addition of one link changes t by $\Delta t = 1/N$, which is the smallest time interval in the problem. Then a single step $\Delta t = 10^{-18}$ from the percolation threshold already gives $S \sim (\Delta t)^\beta \sim 0.1$. The real absence of explosion topples an already established view of explosive percolation. We believe, however, that, thanks to the observed unique properties of this phase transition, our findings make this new class of irreversible systems an even more appealing subject for exploration.

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